Qualifying Exam (Spring 2003): Operations Research

You have 4 hours to do this exam. Do 2 out of problems 1,2,3. Do 2 out of problems 4,5,6. Do 3 out of problems 7,8,9,10,11,12. All problems are weighted equally. On the front page write clearly which seven problems you want graded.

Reminder: This exam is closed notes and closed books.

1). Let x be a nondegenerate BFS for an LP, and suppose that an improving non basic variable enters the basis, and that the minimum ratio test for exiting from the basis has a unique solution. Show that the resulting BFS is nondegenerate.

2). Recall the transshipment (minimum cost network flow) problem:

$$\begin{aligned} \min \quad & z = \textstyle\sum_{i=1}^{n} \sum_{j=1}^{m} c_{ij} x_{ij} & \\ & \textstyle\sum_{j=1}^{m} x_{ij} - \sum_{j=1}^{m} x_{ji} & = b_i \\ & x_{ij} & \geq 0 \end{aligned}$$

Where b_i is the given supply/demand of node i, $\sum_i b_i = 0$. $c_{ij} \geq 0$ is the cost of shipping one unit on arc (i,j), and x_{ij} the variables, are the amounts shipped on the arcs.

(a). Write down the dual of the transshipment problem.

(b). Show that the dual of the transshipment problem is always feasible.

3). Consider the following LP:

$$\begin{aligned} \max \quad & z = 20x_1 + 10x_2 & \\ & x_1 + x_2 & = 150 \\ & x_1 & \leq 40 \\ & x_2 & \geq 20 \\ & x_1, x_2 & \geq 0 \end{aligned}$$

Let the slack (excess) variables be s_i (e_i) for the constraints, and the artificial variables be a_i. The optimal tableau is given below (some entries are missing):

z	x_1	x_2	s_2	e_3	a_1	a_3	RHS
1	0	0		0			1900
0	0	0	-1	1	1	-1	90
0	1	0	1	0	0	0	40
0	0	1	-1	0	1	0	110

(a) Which variables are basic in the current BFS? What is the current B^{-1}?

(b) Complete the optimal tableau.

(c) Find the dual to the original LP and its optimal solution. Make sure to give the optimal value of the dual variables and the objective function.

4). There are 6 nodes on the plane. Five are arranged on a circle and are put at the vertices of a pentagon and the sixth is the center of this circle. There are 10 edges. Five connect the outside nodes and form a pentagon and the other five connect the center with each of the vertices of the pentagon. A particle moves from one node to another according to the following rule: If it is on the outside node, then it moves with probability clockwise, with probability q counterclockwise and with probability r to the center. $(p+q+r=1)$. If the particle is in the center it moves with probability 1/5 to each of the remaining nodes. Let X_n be the position of the particle after n steps.

(a) Find the two-step transition matrix of this Markov chain.

(b) Find the steady state distribution.

5). The participants to the Internet duplicate bridge tournament arrive to the web site at Poisson rate λ. To start a tournament it is necessary to have N pairs. Each new arrival is either paired with the person who arrived just before him or if there are already even number of participants, he waits for the next person to arrive to form a pair. When N pairs are formed, they start a tournament and after that any new arrivals wait to form another N pairs to start another tournament, and so on.

(a) Find the average number of pairs waiting on the web site to start a tournament.

(We count only full pairs, ie., if there are 4 people, then we have two pairs, if there is one person, then we have 0 pairs, and if there are 3 people we have 1 pair, etc.)

(b) What is the average waiting time until the beginning of the tournament for a participant?

6). (A queue with reneging). Customers are arriving to the service station with Poisson rate λ. The service is exponential with parameter μ. If there are n customers already in the system (either waiting or being served) then the newly arrived customer joins the queue with probability $1/(n+1)$.

Find the mean and the variance of the number of customers in the system after it was under operation for a long period of time.

7). Two armies of queens (black and white) *peaceably coexist* on an $N \times N$ chessboard if no two queens from opposing armies can attack one another (two queens from opposing armies can attack one another if they share the same row, column, or diagonal on the chessboard). Formulate an integer program to find the maximum size of two equal-sized peaceably coexisting armies.

8). Let f be a convex function defined on $D \subseteq \mathbb{R}^n$. Suppose that $f(\bar{x})$ is finite. Define $f(x) = +\infty$ if $x \notin D$. Recall that $g \in \mathbb{R}^n$ is called a subgradient of f at $\bar{x}$ if

$$f(y) \geq f(\bar{x}) + g^T(y - \bar{x})$$

for all $y \in \mathbb{R}^n$. Let $\partial f(\bar{x})$ denote the set of subgradients of f at $\bar{x}$.

(a) Prove that $\partial f(\bar{x})$ is bounded if there exists an open neighborhood $U \subseteq D$ of $\bar{x}$ such that f is finite on U.

(b) Show that if $\partial f(\bar{x})$ is not bounded, then there exists a nonzero direction $d \in \mathbb{R}^n$ such that $f(\bar{x}+\epsilon d) = +\infty$ for all $\epsilon > 0$.

9). Consider the max flow problem in a directed graph $D = (N, A)$ with source node s and sink (target) node t, and arc capacities u_{ij}. As usual, let the number of arcs be m, and assume $m \geq n$, the number of nodes.

(a) Suppose a max flow x_{ij} is given. Show how to find a min cut in time $O(m)$.

(b) Suppose that after solving the max flow problem, we realize that the capacity of arc (p,q) was underestimated by k units. (I.e., the true capacity of arc (p,q) is k units more than we used in our solution $u'_{pq} = u_{pq} + k$.) Show that we can find an optimal solution in time $O(mk)$. Hint: use the labeling algorithm.

10). The CONSTRAINED SHORTEST PATH problem is as follows: Given a directed graph $D = (N, A)$ with arc lengths l_{ij}, and arc weights w_{ij}, and integers L and W. (You may assume all lengths and weights are non negative.) Let $s, t \in N$. Is there a path from s to t of length at most L and weight at most W.

...ow that the CONSTRAINED SHORTEST PATH problem is NP-complete. You may use a reduction from ... ARTITION problem: Given integers $a_1, a_2, ..., a_n$, is there a subset $S \subset \{1, ..., n\}$, such that $\sum_{i \in S} a_i = \sum_{i \notin S} a_i$.

(b) Now consider a special case of the CONSTRAINED SHORTEST PATH problem in which all weights of arcs are 1 ($w_{ij} = 1$ for all arcs). Show that this problem can be solved in polynomial time. Hint: Think Belman Ford.

11). Consider a small bank with two tellers. Teller 1 deals only with business accounts, while Teller 2 deals exclusively with general accounts. Each teller has his/her own separate queue. Clients arrive at the bank according to a Poisson process at a rate of one customer every 5 minutes. Of the clients, 33% are business accounts and the rest are general accounts. A business-account client take 7 ± 5 minutes to complete service at his/her teller (*i.e.*, uniformly distributed in the range [2 , 12] minutes), and general-account clients 3 ± 2 minutes.

(a) Identify the state variables and events you would need to implement an event-oriented simulation model of this facility. For each event, write pseudo-code that shows the actions that the event undertakes. Note that you are **not** required to write a full-blown program, but simply a pseudo-code description of what each event does, together with an explanation of the variables and data structures your events use. Assume that you have at your disposal a function *Random*( ) which returns to you uniformly distributed random variates in the range (0.0 , 1.0), but no other random number generation capabilities. Also assume that you have available a function *Schedule*(*etype* , *t*) which schedules an event of type *etype* for time instant *t*.

(b) Carefully describe how you would go about gathering statistics so that the simulation program to be implemented calculates, at the end of its run, the mean number of (business) customers in the queue for Teller 1. Assume that the program will be encoded in some high-level language such as *C* or *C++*, and not in general purpose simulation language such as *SIMSCRIPT*. Thus, you may not assume that any statistics-gathering capabilities are available to you in the programming language.

12). Consider a random variable X which has a triangular distribution with *pdf*

$$f(x) = \begin{cases} x, & \text{if } 0 \le x \le 1 \\ 2 - x, & \text{if } 1 < x \le 2 \\ 0, & \text{otherwise} \end{cases}$$

Generate the first random variate value from the distribution for X. You may use any method you choose, but you must state what that method is, and clearly show and explain each step involved in the development of your final answer.

You will need to generate uniformly distributed random variates in the range (0.0 , 1.0). Use the (Mixed) *LCG* (Linear Congruential Generator)

$$X_{n+1} = (\, 7X_n + 29 \,) \; mod \; 100$$

with intial seed value $X_0 = 37$.