Qualifying Exam (Fall 1999)
Operations Research

You have 4 hours to do this exam. Do 2 out of problems 1,2,3. Do 2 out of problems 4,5,6. Do 3 out of problems 7,8,9,10,11,12. All problems are weighted equaly. On the front page write clearly which seven problems you want graded.
Reminder: This exam is closed notes and closed books.

1). Consider the set of constraints to a Linear Programming problem: $P = \{Ax \geq b,\ x \geq 0\}$
(a). Describe a method to detect whether there is a point $x' \in P$ feasible to the constraints, in which $x_i > 0$.
(b). Describe a method to find a feasible point $x \in P$ maximizing the number of vairables that are strictly greater than zero.

2). Consider the following LP:

$$\begin{aligned} \max \quad & z = 7x_1 + 3x_2 + 2x_3 & \\ & 4x_1 + x_2 + x_3 & \leq 18 \\ & 3x_1 + 2x_2 + x_3 & \leq 14 \\ & x_1, x_2, x_3 & \geq 0 \end{aligned}$$

Let the slack variables be s_1 and s_2 for the constraints. After some number of pivots, the following tableau was obtained (some of the entries are missing):

	z	x_1	x_2	x_3	s_1	s_2	RHS
z	1	0			1	1	
x_1	0	1			1	-1	
x_3	0	0			-3	4	

(a). Which variables are basic in the current BFS? What is the current B^{-1}?
(b). Could the current BFS be obtained from the original BFS (with s_1 and s_2 basic) using a single pivot? Explain.
(c). Determine the values of the missing entries in the tableau.
(d). Is the current tableau optimal? If so, is the optimal solution unique?

3). Each day a company produces capacitors during three shifts: 8am-4pm, 4pm-midnight, midnight-8am. The hourly salary paid to emloyees on each shift, the price charged for each capacitor made during each shift, and the average number of defects in each capacitor produced during a given shift are given in the table below. Each of the company's 25 workers must be assigned to at most one of these three shifts. A worker produces 10 capacitors during a shift. (An unassigned worker produces no capacitors, and does not get paid.) Due to machinary limitations, no more than 10 workers can be assigned to any one shift. Each day at most 250 capacitors can be sold, and the average number of defects per day cannot exeed three.

shift	hourly salary	defects per capacitor	price
8am-4pm	\$12	4	\$18
4pm-midnight	\$16	3	\$22
midnight-8am	\$20	2	\$24

Formulate an LP to maximize daily profit (sales revenue minus labor cost). Make sure to clearly define your variables!

4). Customers arrive to the restaurant according to a nonhomogeneous Poisson process with intensity which increases at a constant rate from 0 to 20/hour from 11 a.m until 12 p.m., then it stays constant (20/hour) from 12 p.m. until 2 p.m. during the next two hours it decreases at a constant rate from 20 to 0/hour. Write an expression for the probability that the number of customers which arrives from 11 a.m. until 1 p.m is the same as the number which arrives from 1 p.m. until 4 p.m.

5). A fugitive moves about three cities, Houston, Galveston and Sugarland, to escape a bounty hunter. Initially the fugitive is in Houston and the bounty hunter is in Galveston. The fugitive and bounty hunter move independently of each other and each follows the respective transition probability matrix of a markov chain:

$$P_{\text{fugitive}} = \begin{array}{c} \\ \text{Houston} \\ \text{Galveston} \\ \text{Sugarland} \end{array} \begin{bmatrix} & .4 & .6 \\ .7 & & .3 \\ .1 & .9 & \end{bmatrix} \qquad P_{\text{bounty hunter}} = \begin{array}{c} \\ \text{Houston} \\ \text{Galveston} \\ \text{Sugarland} \end{array} \begin{bmatrix} .3 & .2 & .5 \\ .8 & .1 & .1 \\ .5 & .2 & .3 \end{bmatrix}$$

We assume that each party makes the move (transition) at the end of each day. When the two are in the same city on a given day there is a 50 percent chance the fugitive will be captured. Write equations whose solutions will give the expected time for the bounty hunter to catch the fugitive. (No need to solve the equations.)

6). A job shop consists of three machines and two repairmen. The amount of time a machine works before breaking down is exponentially distributed with a mean 10. If the amount of time it takes a single repairman to fix a machine is exponentially distributed with mean 8, then
(a) what is the average number of machines not in use?
(b) what proportion of time are both repairmen busy?

7). (a) Explain what is meant by "Variance Reduction Techniques" as a simulation methodology. What is being reduced exactly, and in what way does that make for a more efficient simulation? Be as precise as you can in your answers.
(b) Discuss how the use of Variance Reduction Techniques impacts your freedom of choice as to the method of random number generation you can use. Cite an example of a random number generation technique that is not suitable for use with Variance Reduction, and explain precisely why it is not suitable.

8). (a) Explain why the Method of Batches ("Method of Batched Means") is considered preferable as a simulation methodology to the Method of Independent Replications. Mention all relevant aspects of the Method of Batches you can think of, and explain the advantage each gives over the alternative method.
(b) What is the central problem you would have to worry about if you were to use Batched Means instead of Independent Replications (i.e., something that would not be an issue if you were to use Independent Replications)? In what way exactly could that problem distort your the analysis of your simulation results if you were not careful about making sure it does not occur? Be as precise as you can in your answers.

9). Let S be a set of n points in the plane.
(a) How efficiently can one compute a triangulation, $\mathcal{T}$, of the convex hull of S. (Here, *any* triangulation is fine.) Give the best upper and lower bounds that you can, and briefly justify your answer.
(b) Now suppose that the points S are the vertices of a simple polygon, P, given in order about the polygon. Again, the goal is to triangulate the convex hull of S (both that portion that lies inside P and that portion that lies outside P). Answer now the same questions as in part (a).
(c) Now suppose that we want to preprocess a triangulation, $\mathcal{T}$, of S to be able to answer the following type of query efficiently: Given a query line, ℓ, determine the number of triangles of $\mathcal{T}$ that intersect ℓ. Explain how to answer these queries in time $O(\log n)$. What is the preprocessing time for your method? What is the storage space?

10). You have just planted in your yard a set of n new tree seedlings. (Each is so tiny, you can think of it as a point.) You have only one sprinkler, which sprays water over a (circular) disk of radius r; you can adjust r by adjusting the water pressure up or down. Your goal is to place the sprinkler in such a way that you can water all of the seedlings, while keeping the water pressure as low as possible. Interpret the problem in terms of the "lifted" points onto a paraboloid. How efficiently can you determine where to place it? Sketch an algorithm and analyze its complexity.

11). (a) In a minimum cost flow problem, show that an optimal flow (with no negative cycles in the residual network) satisfies the reduced cost optimality conditions.
(b) Show that the complementary slackness conditions imply the reduced cost optimality conditions.

12). Suppose you are given a nonintegral optimal solution to a minimum cost flow problem with integral data. Propose a method for converting this solution into an integer optimal solution. Your method should maintain optimality of the solution at every step.