Qualifying Exam (Spring 2004): Operations Research

You have 4 hours to do this exam. **Reminder:** This exam is closed notes and closed books.

Do 2 out of problems 1,2,3.

Do 2 out of problems 4,5,6.

Do 3 out of problems 7,8,9,10,11,12,13,14.

All problems are weighted equally. On this cover page write which seven problems you want graded.

problems to be graded:

Academic integrity is expected of all students at all times, whether in the presence or absence of members of the faculty. Understanding this, I declare that I shall not give, use, or receive unauthorized aid in this examination.

Name (PRINT CLEARLY), ID number

Signature

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Reminder: This exam is closed notes and closed books.

1). Consider the LP $\min\{cx \mid Ax = c,\ x \geq 0\}$, where A is a square symetric n by n matrix ($A = A^t$). Prove that if there exists a point x_0 such that $Ax_0 = c$ and $x_0 \geq 0$, then x_o is an optimal solution.

2). Consider the LP $\min\{cx \mid Ax = b,\ x \geq 0\}$, and let $\min\{w = \mathbf{1}x_a \mid Ax + Ix_a = b,\ x, x_a \geq 0\}$ be the phase I problem ($\mathbf{1}$ is a vector of ones, x_a is a vector of artificial variables). In class we said that when solving the phase I problem, we can discard any artificial variable from the tableau whenever it leaves the basis. Let w^1 be the value of w at the termination of phase 1 when artificial variables are discarded (dropped) any time they leave the basis.

Now consider solving the phase I LP without discarding any variables, and let w^2 be the optimal value of the objective function.

(a). Prove that if $w^1 > 0$ then $w^2 > 0$.

(b). Prove that $w^1 = 0$ if and only if $w^2 = 0$.

(c). Is w^1 always equal w^2? Give a proof or a brief explanation why not.

3). A bicycle manufacturer has 2 plants in which it manufactures bicycles. The company has four customers, whose demand it must meet. Costomer A requires 500 bicycles, customer B 300 bicycles, customer C 1,000 bicycles and customer D 200 bicycles. Shipping costs of bicycles are given in the table below:

	to A	to B	to C	to D
From plant 1	\$ 1	\$ 2	\$ 3	\$ 4
From plant 2	\$ 5	\$ 3	\$ 2	\$ 1

Each bicycle must be machined and assembled. The cost, requirements and labor availabilities in the two plants are as follows:

	Hours per bicycle	cost per hour	Hours available
Plant 1 machining	0.1	\$ 16	120
Plant 1 assembling	0.2	\$ 12	260
Plant 2 machining	0.12	\$ 15	120
Plant 2 assembling	0.22	\$ 11	240

The company wants to minimize its total costs to fill the demands, and asks that you formulate an LP to do so. Make sure to clearly define the variables used in your formulation.

4). Let the numbers of children in families $X_1, X_2, \ldots$ be i.i.d. with $P(X_1 = k) = q_k$, $k = 0, 1, \ldots$, and mean $a = E(X_1)$. Find the probability that a randomly selected child is jth born and has exactly l younger siblings.

5). Suppose that 2/3 of the trucks on a highway are followed by a car but only 1/5 of the cars are followed by a truck. What fraction of vehicles on the highway are cars (busses are not allowed to the highway).

6). The arrival process of customers at a taxi stand is Poisson at rate λ, and the arrival process of taxis at the stand is Poisson at rate μ. Arriving customers who find taxis waiting leave immediately in a taxi.

Arriving taxis who find customers waiting leave immediately with one customer each. Otherwise, customers will queue up to a limit of c customers, i.e., arriving customers who find c other customers waiting leave immediately without a taxi. Similarly, taxis queue up to a limit of t taxis. Find the average number of customers and the average number of taxis waiting at the stand.

7). In a city, population is concentrated in I districts within the city and district i contains p_i people, $i = 1, \ldots, I$. There are J potential sites to build hospitals. Let $d_{ij} \geq 0$ denote the distance from the center of district i to site j, $i = 1, \ldots, I, j = 1, \ldots, J$. Each district has to be assigned to exactly one hospital. Due to legal regulations, either sites 1 and 2 together or sites 3 and 4 together have to be used. Finally, assume that it costs f_j dollars to build a hospital at site j and that the annual cost of operation is linear in the number of people in the serviced districts (c_j dollars per person), $j = 1, \ldots, J$. The city's annual budget for health services is B dollars. Formulate an integer programming problem to minimize the maximum distance from a district to its assigned hospital.

8). Consider the following equality constrained quadratic optimization problem:

$$\begin{array}{ll} \min_x & c^T x + \frac{1}{2} x^T Q x \\ \text{subject to} & Ax = b, \end{array}$$

where $c \in \mathrm{R}^n$, $x \in \mathrm{R}^n$, $b \in \mathrm{R}^m$, $A \in \mathrm{R}^{m \times n}$, and $Q \in \mathrm{R}^{n \times n}$ is a symmetric matrix. Suppose that the problem has a feasible solution and that for all nonzero vectors $z \in \mathrm{R}^n$ in the null space of A (i.e., $Az = 0$), we have $z^T Q z > 0$.

(a) Show that the optimization problem has a unique global optimum x^*. Compute x^*.

(b) Let $\{x_n\}$ be a sequence of feasible solutions such that $\|x_n\| \to \infty$ as $n \to \infty$. Show that the objective function along this sequence goes to $+\infty$.

9). An entrepreneur faces the following problem: In each of T periods, he can buy, sell, or hold for later sale, a certain commodity, subject to the following constraints. In each period i he can buy at most α_i units of the commodity, can hold over at most β_i units of the commodity for the next period, and must sell at least γ_i units (perhaps due to prior agreements). The entrepreneur *cannot* sell the commodity in the same period in which he buys it. Assuming that p_i, w_i, and s_i denote the purchase cost, inventory carrying cost, and selling price per unit in period i, what buy-sell policy should the entrepreneur adopt to maximize total profit in the T periods? Formulate as a min cost flow problem. (We may assume that $\gamma_1 = 0$ otherwise the problem is not feasible.)

10). (a) We defined a *tree* as a connected acyclic graph. We had also shown in class that a tree on n nodes contains $n-1$ edges. Show that a graph G is a tree if and only if it is acyclic and whenever an edge is added between two arbitrary nodes in G (which do not have an edge between them in G) the resulting graph G' has exactly one cycle.

(b) (unrelated to part (a)) A *forest* is an undirected (simple) graph with no cycles. Prove that a forest containing k connected components and n nodes has $n-k$ edges.

11). Consider a customer service call center which is open to incoming calls 24 hours a day, seven days a week. The rate of incoming calls varies on an hourly basis over a 24-hour period (*i.e.*, each hour in the 24 has a different rate of incoming calls). This hourly pattern of incoming calls then repeats itself for the next 24 hours.

We are interested in developing a simulation studying of this system to determine adequate staffing levels that ensure a reasonable level of customer service, in terms of the amount of time an incoming call is placed on hold awaiting service (mean time, 99th. percentile, and so on).

(a) In what sense, and under what conditions, may this system be said to achieve steady state; and in what sense my it be said **not** to achieve steady state?

Please be as precise as you can in your answer, and carefully explain any assumptions you are making about the characteristics of the system/model.

(b) Carefully explain the simulation methodology you would implement to study this system. Be precise and detailed in your answer, making sure you address the following issues, and any others you deem relevant:

(i) What would the characteristics of the model have to be for you to be able to get away with a single run of the model? Under what circumstances, on the other hand, would you have to undertake multiple, independent runs?

(ii) What time frame would you collect your performance metrics over? Hourly? Daily (24-hour period)? Both? Some other time frame? Why?

(iii) What would the characteristics of the model have to be for you to be able to start collecting performance metrics right from the first 24-hour period of a run? Under what conditions, on the other hand, would you need to determine a "transient" phase for the model?

12). Consider the following *CDF*

$$F(x) = \begin{cases} (2 + 3x - x^3)\,/\,4\,, & \text{if} \quad -1 \le x \le 1 \\ 0\,, & \text{otherwise} \end{cases}$$

Give **two** different methods by which we can generate independent variates from this distribution. For each method, be as detailed and precise as you can: your answers should come as close to providing a detailed, specific algorithm for this particular distribution as you can, rather than a general description of the method. Show rough plots of the *CDF* and/or *pdf* if you think this will be helpful.

13). (a). Given a set $\mathcal{C}$ of n unit disks in the plane (arbitrarily overlapping), explain briefly how to build a data structure to support efficient queries of the form: For a query point q, determine if q lies inside some disk of $\mathcal{C}$ and, if so, report one such disk. State the preprocessing time, storage space, and query time.

(i). Preprocessing time is $O(\qquad)$

(ii). Storage space (memory usage) is $O(\qquad)$

(iii). Query time is $O(\qquad)$

(b). Given a set P of n points in the plane, explain briefly how to build a data structure to support efficient queries of the form: For a query segment σ, known to be axis-parallel (i.e., horizontal or vertical), and a number $\delta > 0$ (given as part of the query), determine if there exists some point $p \in P$ whose Euclidean distance from σ is at most δ, and, if so, report one such point. (The distance from p to σ is defined to be $\inf_{q\in\sigma} d_2(p, q)$, where $d_2(\cdot,\cdot)$ denotes Euclidean distance.) State the preprocessing time, storage space, and query time.

(i). Preprocessing time is $O(\qquad)$

(ii). Storage space (memory usage) is $O(\qquad)$

(iii). Query time is $O(\qquad)$

14). Let S be a set of n noncrossing line segments in the plane. (Two segments are allowed to share endpoints, but are not allowed to intersect at a point that is interior to either segment.)

(a). How efficiently can one compute a triangulation, $\mathcal{T}$, of the set S within the convex hull of S? (In other words, we want a decomposition of the convex hull of S into triangles, such that every segment of S appears as an edge of some triangle.) Here, *any* triangulation is fine. Give the best upper and lower bounds that you can, and briefly justify your answer.

(b). Now suppose that the union of the segments S is connected. Answer now the same question as in part (a): How efficiently can one triangulate the convex hull of S? Explain briefly.

(c). Now suppose that we want to preprocess a triangulation, $\mathcal{T}$, of S to be able to answer the following type of query efficiently: Given a query line, ℓ, determine the number of triangles of $\mathcal{T}$ that intersect ℓ. Explain briefly how to answer these queries in time $O(\log n)$, after an appropriate preprocessing step. What is the preprocessing time for your method? What is the storage space?