APPLIED MATHEMATICS and STATISTICS

DOCTORAL QUALIFYING EXAMINATION

in COMPUTATIONAL APPLIED MATHEMATICS

Spring 2000

(CLOSED BOOK EXAM)

This is a two part exam.
In part A, solve 4 out of 5 problems for full credit.
In part B, you must also solve 4 out of 5 problems for full credit.

Indicate below which problems you have attempted by circling the appropriate numbers:

Part A:	1	2	3	4	5
Part B:	6	7	8	9	10

NAME ________________________________

Start each answer on its corresponding question page. Print your name, and the appropriate question number at the top of any extra pages used to answer any question. Hand in all answer pages.

Date of Exam: Tues., Jan. 25, 2000

Time: 12:30 – 4:30 PM

Place: SBS, N-101

A1. Consider the coupled first-order differential equations

$$\frac{dx}{dt} = -2y + yz, \qquad \frac{dy}{dt} = x(1-z), \qquad \frac{dz}{dt} = xy.$$

(a) Find all critical points of this system.

(b) Discuss the stability of the linearized equations at the origin (0,0,0) using Liapunov's function.

A2.

(a) Describe the role of the Green's function in the solution of self-adjoint Sturm-Liouville Problems.

(b) Construct the Green's function for the differential operator

$$-\frac{d^2}{dx^2}$$

corresponding to the boundary conditions $y'(0) = y'(1) = 0$.

A3. Find the solution of the initial/boundary-value problem

$$\begin{aligned} &u_t(x,t) = u_{xx}(x,t) + x && \text{for } x \in (0,1), t > 0,\\ &u(x,0) = x(1-x^2)/6 + \sin(2\pi x) && \text{for } x \in (0,1),\\ &u(0,t) = 0 && \text{for } t > 0,\\ &u(1,t) = 0 && \text{for } t > 0. \end{aligned}$$

A4. Let p, $q \geq 0$ and define

$$N(x,t) = \begin{cases} x/t & \text{if } -\sqrt{pt} < x < \sqrt{qt},\\ 0 & \text{otherwise} \end{cases}$$

for $x \in R$ and $t > 0$.

(a) Show that $u = N$ is a weak solution of Burgers' equation,

$$u_t + \left(\frac{u^2}{2}\right)_x = 0.$$

(b) Show that, for all $t > 0$, the quantity

$$\inf_{x \in R} \int_{-\infty}^{x} N(z,t)\, dz$$

is independent of t. Determine its value.

A5. Let $w = f(z)$ be analytic for $|z| < 1$. If $0 \le r < 1$, prove that the image in the w-plane of the circle $|z| = r$ has length L, such that

$$L \ge 2\pi r |f'(0)| \ .$$

[The length L is given by $L = \int |dw|$.]

B6. For the matrix

$$A = \begin{pmatrix} 5.2 & 0.6 & 2.2 \\ 0.6 & 6.4 & 0.5 \\ 2.2 & 0.5 & 4.7 \end{pmatrix}$$

compute an upper bound for $\mathrm{cond}_2(A)$, using estimates of the eigenvalues by the method of Gershgorin.

B7. Suppose that $f(\xi) = 0$, where the function f has a continuous second derivative in the interval $I := [\xi - k, \xi + k]$, and that there exists a positive constant A such that

$$\frac{|f''(x)|}{|f'(y)|} \le A, \quad \forall \quad x, y \in [\xi - k, \xi + k].$$

Show that if $|\xi - x_0| \le h$, where h is the smaller of k and $1/A$, the iteration x_n obtained by Newton's method converges quadratically to ξ.

B8. Suppose the real matrix A is symmetric with positive diagonal entries. Then the Jacobi method for solving the system $Ax = y$ is convergent if and only if both A and $2D - A$ are positive definite, where $D = diag(a_{11}, \cdots, a_{nn})$.

B9. Consider the fundamental polynomial interpolation formula specialized to degree 3,

$$f(x) = p_3(x) + f[x_0, \ldots, x_3, x]\psi_3(x) \ .$$

(a) Derive a finite difference approximation to $f'''(a)$ using the points $x_0 = a$, $x_1 = a - h$, $x_2 = a + h$, $x_3 = a + 2h$;

(b) What is the leading order error term (in h)?

B10. Find a rule of the form

$$\int_{-1}^{1} f(x)dx \approx A_0 f(-1) + A_1 f(x_1) + A_2 f(1)$$

which is exact for all polynomials of degree ≤ 3. (This is a particular example of the more general Lobatto's rule.)