APPLIED MATHEMATICS and STATISTICS

DOCTORAL QUALIFYING EXAMINATION

in COMPUTATIONAL APPLIED MATHEMATICS

Spring 2001

(CLOSED BOOK EXAM)

This is a two part exam.
In part A, solve 4 out of 5 problems for full credit.
In part B, you must also solve 4 out of 5 problems for full credit.

Indicate below which problems you have attempted by circling the appropriate numbers:

Part A:	1	2	3	4	5
Part B:	6	7	8	9	10

NAME ____

Start each answer on its corresponding question page. Print your name, and the appropriate question number at the top of any extra pages used to answer any question. Hand in all answer pages.

Date of Exam: Fri., Jan. 26, 2001

Time: 12:00 – 4:00 PM

Place: MATH, 1-122A

A1. Let

$$\frac{dx}{dt} = -\nabla\ \psi(x), \quad x \in R^n,$$

where ψ is C^2. Let x_0 be an isolated minimum of ψ. Prove that x_0 is an asymptotically stable equilibrium.

A2. Consider the coupled, first-order differential equations

$$\frac{dx}{dt} = 14\,x - \frac{1}{2}x^2 - x\,y\ ,$$
$$\frac{dy}{dt} = 16\,y - \frac{1}{2}y^2 - x\,y\ .$$

a) Find all critical points of this system.

b) Discuss the stability of the linearized equations at two of the critical points (other than the point (0,0)) found in part (a). One of these critical points must be away from the coordinate axes.

c) Sketch the phase portrait of this system.

A3. Consider a solution u of the following initial-value problem for a scalar conservation law:

$$u_t + f(u)_x = 0 \qquad \text{for } x \in R \text{ and } t > 0,$$
$$u(x,0) = u_0(x) \qquad \text{for } x \in R.$$

Here f is continuously differentiable and u_0 is continuous. Assume that u is continuous on $R \times [0, \infty)$ and continuously differentiable in $R \times (0, \infty)$. Define U and U_0 by

$$U(x,t) = \int_0^x u(y,t)\,dy - \int_0^t f(u(0,s))\,ds,$$
$$U_0(x) = \int_0^x u_0(y)\,dy$$

for $x \in R$ and $t \geq 0$.

(a) Show that U satisfies the initial-value problem

$$U_t + f(U_x) = 0 \qquad \text{for } x \in R \text{ and } t > 0,$$
$$U(x,0) = U_0(x) \qquad \text{for } x \in R.$$

(b) Use the method of characteristics to reduce the initial-value problem of part (a) to the initial-value problem for a system of ordinary differential equations. (You do not have to solve these equations.)

Hint: Recall that, for the first-order scalar PDE

$$F(x_1, \ldots, x_n, z, p_1, \ldots, p_n) = 0,$$

the characteristic equations are $\dot{x}_j = F_{p_j}$ and $\dot{p}_j = -F_{x_j} - F_z p_j$ for $j = 1, \ldots n$, together with $\dot{z} = \sum_{k=1}^{n} p_k F_{p_k}$.

A4. Let Ω be a bounded, open, connected subset of R^n with a smooth boundary, and let α be a constant. Consider the boundary-value problem

$$\begin{aligned} \Delta u &= f \quad \text{in } \Omega, \\ \frac{du}{dn} + \alpha u &= g \quad \text{on } \partial\Omega. \end{aligned}$$

Show that there is at most one solution $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ if $\alpha > 0$.

A5. A function $f(z)$ is said to be *univalent* in a domain D if it is analytic in D and assumes no value more than once in D; that is, $f(\xi) = f(\eta), \quad \xi, \eta \in D$, implies $\xi = \eta$. Prove that if $f(z)$ is univalent in D, then the derivative $f'(z)$ does not vanish in D.

B6. Consider the matrices

$$A = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad E = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & \epsilon \\ 0 & \epsilon & 0 \end{pmatrix}$$

Let λ_1 denote the smallest eigenvalue of A and μ_1 denote the smallest eigenvalue of $A+E$.

a) Using the Bauer-Fike theorem, provide either an l_∞ or l_1 norm bound for the difference $|\lambda_1 - \mu_1|$.

b) What does Gerschgorin's theorem say about the location of μ_1?

c) What is the best possible improvement for the estimate of the location of μ_1 using Wilkinson's correction method?

B7. Consider the system $Ax = b$ with

$$A = \begin{pmatrix} 1 & a & a \\ a & 1 & a \\ a & a & 1 \end{pmatrix}.$$

a can be either positive or negative.

a) For what range of values of a will the Jacobi method always converge?

b) For what range of values of a will the SOR method always converge?

(*Hint*: use the fact that $(x_1+x_2+x_3)^2 \geq 0$ to find a bound for $2|x_1x_2+x_2x_3+x_3x_1|$.)

c) What is the value of $\rho(H_J)$ for this system?

B8. Consider the following theorem on the local convergence of Newton's method:
Assume that $F : R^n \to R^n$ is differentiable at each point of an open neighborhood of a solution x^* of $Fx = 0$; that F' is continuous at x^*; and that $F'(x^*)$ is nonsingular. Then x^* is a point of attraction of Newton's method and Newton's method has a superlinear convergence rate. Moreover, if

$$||F'(x) - F'(x^*)|| \leq \beta||x - x^*||$$

for all x in some open neighborhood of x^*, then the convergence rate of Newton's method is quadratic.
Consider Newton's method applied to the problem $Fx = 0$ in R^1, where $Fx = x + x^{1+\alpha}$ with $0 < \alpha < 1$. In the light of this theorem:

a) What value x^* solves $F(x) = 0$?

b) Is x^* a point of attraction? For either conclusion, support your answer.

c) What is the rate of convergence of Newton's method to x^*? Again, support your answer.

B9. The Chebyshev polynomial of degree n is defined through trigonometric function

$$T_n(\cos\theta) = \cos(n\theta)$$

with

$$T_0(x) = 1, \qquad T_1(x) = x \ .$$

(a) Find the recursive relation to derive $T_n(x)$ from lower order Chebyshev polynomials.

(b) Prove that Chebyshev polynomials $T_m(x)$ and $T_n(x)$ $(n \neq m)$ are orthogonal in the interval $[-1, 1]$.

(c) Expand the function $p(x) = x^3$ into Chebyshev polynomials.

B10. A function $f(x)$ can be interpolated through Newton's forward divided difference

$$f(x) \approx f[x_0] + f[x_0, x_1](x - x_0) + \cdots f[x_0, x_1, \cdots, x_n](x - x_0) \cdots (x - x_{n-1}) \ .$$

Prove that the truncation error in this interpolation is

$$e(x) = f[x_0, \cdots, x_n, x](x - x_0)(x - x_1) \cdots (x - x_n) \ .$$

Show that the interpolation degenerates into a Taylor expansion if $x_0 = x_1 = \ldots = x_n$. What is the truncation error in the Taylor expansion?