APPLIED MATHEMATICS and STATISTICS

DOCTORAL QUALIFYING EXAMINATION

in COMPUTATIONAL APPLIED MATHEMATICS

Spring 2002

(CLOSED BOOK EXAM)

This is a two part exam.
In part A, solve 4 out of 5 problems for full credit.
In part B, you must also solve 4 out of 5 problems for full credit.

Indicate below which problems you have attempted by circling the appropriate numbers:

Part A:	1	2	3	4	5
Part B:	6	7	8	9	10

NAME ______________________________

Start each answer on its corresponding question page. Print your name, and the appropriate question number at the top of any extra pages used to answer any question. Hand in all answer pages.

Date of Exam: Fri., Jan. 25, 2002

Time: 12:00 – 4:00 PM

Place: Math, 1-122A

A1. Let $A(t)$ be an $n \times n$ matrix with continuous periodic elements. Suppose that $\dot{x} = A(t)x$ is stable at the origin. Prove that the nonlinear system

$$\dot{x} = A(t)\, x + g(x,t) \; , \qquad x(0) = c \; ,$$

is also stable at the origin for every $g(x,t)$ satisfying the following condition:
There exist constants δ and η so that

$$\int_0^\infty \frac{\| \; g(x(s),s) \; \| \; ds}{\| \; x(s) \; \|} \leq \eta$$

for every continuous function $x(s)$ such that $\| \; x(s) \; \| < \delta$.

A2. Let $f(x)$ be a continuous function of the real variable x. Using the infinite series

$$y(x) = \sum_{0}^{\infty} \epsilon^n \, y_n(x) \qquad (*)$$

as a solution of the initial-value problem,

$$y'' = \epsilon f(x) y(x) \; , \quad y(0) = 1 \; , \quad y'(0) = 1 \; ,$$

show:

a)

$$y_0(x) = 1 + x \; , \qquad y_n(x) = \int_0^x dt \int_0^t f(s) y_{n-1}(s) ds \; , \qquad n \geq 1 \; ;$$

b) the infinite series (*) is uniformly convergent for all closed intervals, and provides approximate solutions of the initial-value problem

$$y'' = f(x) y(x) \; , \qquad y(0) = 1, \quad y'(0) = 1 \; .$$

A3. Consider the initial-value problem

$$u_t + uu_x + u = 0,$$
$$u|_{t=0} = u_0,$$

for the function $u : \mathbf{R} \times [0, \infty) \to \mathbf{R}$. Here $u_0 : \mathbf{R} \to \mathbf{R}$ is specified.

(a) Show that the solution is given implicitly by the equation

$$u = e^{-t} u_0(x + u).$$

(b) Assuming that u_0 is smooth and $|u_0'(x)| < 1$ for all $x \in \mathbf{R}$, show that the solution u is smooth.

A4. Let $\Omega = (0,1) \times (0,1)$ be the unit square. Consider the heat equation

$$u_t = \Delta u \qquad \text{for } (x,y) \in \Omega \text{ and } t > 0$$

subject to the boundary conditions

$$\begin{aligned} u(0,y,t) &= u(1,y,t) && \text{for } y \in [0,1] \text{ and } t > 0, \\ u(x,0,t) &= 0 && \text{for } x \in [0,1] \text{ and } t > 0, \\ u(x,1,t) &= 0 && \text{for } x \in [0,1] \text{ and } t > 0. \end{aligned}$$

(a) Find the solution of the initial-value problem with initial condition

$$u(x,y,0) = f(x,y) \qquad \text{for } (x,y) \in \overline{\Omega},$$

where f is a continuous function defined on $\overline{\Omega}$ that satisfies $f(0,y) = f(1,y)$, $f(x,0) = 0$, and $f(x,1) = 0$ for all $x \in [0,1]$ and $y \in [0,1]$.

(b) Suppose that

$$f(x,y) = \begin{cases} \sin(\pi y) & \text{for } x \in [0,1/2] \text{ and } y \in [0,1], \\ -\sin(\pi y) & \text{otherwise.} \end{cases}$$

Compute the solution of the initial-value problem given by the formula derived in part (a).

(c) Does the solution computed in part (b) satisfy the initial conditions? Explain.

A5. If the power series $p(z) = a_0 + a_m z^m + a_{m+1} z^{m+1} + \dots\,, \quad (a_m \neq 0)$, converges in a neighborhood of the origin, prove that, for sufficiently small $\epsilon > 0$, there exists a point z_0, $|z_0| = \epsilon$, such that $|p(z_0)| > |a_0|$. Use this result to prove the maximum principle.

B6. Consider a 2 dimensional ellipse on R^3 whose semi axes are

$$v_1 = (2, 0, 1)^T , \qquad \text{and} \qquad v_2 = (-1/\sqrt{5}, 0, 2/\sqrt{5})^T .$$

a) Find the linear map from R^3 to R^3 which maps the unit ball in R^3 into the ellipse, and maps $(1, 0, 0)^T$ and $(0, 1/\sqrt{2}, 1/\sqrt{2})^T$ to v_1 and v_2, respectively. Write the linear map as a 3 by 3 matrix. (Here $()^T$ denotes transpose.)

b) What is the 2-norm condition number of the matrix?

B7.

a) Suppose $U = (u_{ij})$ is an $m \times m$ invertible, upper triangular matrix, and x and b are vectors in C^m. Fill in the blank line in the following algorithm to solve $Ux = b$ for x by back-substitution. (Make sure your indicies are correct.)

for $j = m$ **to** 1
    $b = b_j$
    for $k = j + 1$ **to** m
        (   ) \\ Fill in this line to update b
    $x_j = b/u_{jj}$

b) Let A be an invertible $m \times m$ matrix with the factorization $A = LU$, where L is a unit lower triangular matrix and U is upper triangular. Let **bsub**$(arg_1, arg_2, arg_3, arg_4)$ be a subroutine that only performs back-subsititution. Given the factor matrices L and U and the m-vector b, outline an algorithm for solving $Ax = b$ that would only use the subroutine **bsub**$(\ldots)$, i.e. in your algorithm all floating point operations would be performed only by the subroutine **bsub**$(\ldots)$. You do not have to write specific lines of code, but indicate, in sequential order, what specific goal would be accomplished by each block of code that would be writen. Your algorithm should also identify what the 4 arguments of **bsub**() are and, for any call to **bsub**() made by your algorithm, you should indicate the specific content of each argument.

B8. A polynomial interpolation $p(x)$ to the function $f(x)$ on $n+1$ points $x_0, x_1, x_2, \cdots, x_n$ can be written in the form of the Newton divided difference

$$p_N(x) = f[x_0] + f[x_0, x_1](x - x_0) + \cdots + f[x_0, x_1, \cdots, x_n](x - x_0) \cdots (x - x_{n-1})$$

and the Lagrangian form

$$p_L(x) = L_0(x)f(x_0) + L_1(x)f(x_1) + \cdots L_n(x)f(x_n) \ .$$

a) What properties must the Lagrangian functions $L_i(x)$ have in order to satisfy the node condition $p(x_i) = f(x_i)$. Show that the function

$$L_i(x) = \frac{\psi(x)}{(x - x_i)\psi'(x_i)}, \qquad i = 0, 1, \cdots, n \qquad \psi(x) = (x - x_0)(x - x_1) \cdots (x - x_n)$$

satisfy such conditions.

b) Prove that Newton interpolation and Lagrangian interpolation are identical, i.e.

$$p_N(x) \equiv p_L(x) \ .$$

c) Prove that

$$f[x_0, x_1, \cdots, x_n] = \sum_{i=0}^{n} \frac{f(x_i)}{\psi'(x_i)} \ .$$

d) Calculate

$$\lim_{x_1 \to x_0} f[x_0, x_1, \cdots, x_n] \qquad \text{and} \qquad \lim_{x_i \to x_0, i=0,1,\cdots,n} f[x_0, x_1, \cdots, x_n] \ .$$

B9. For numerical approximation of the integral

$$I = \int_{-1}^{1} f(x)dx$$

use

$$I = C_0 f(x_0) + C_1 f(x_1) + C_2 f(x_2)\ .$$

Choose appropriate x_0, x_1, x_2, C_0, C_1 and C_2 so that the approximation is EXACT if $f(x)$ is a polynomial with degree less than 6.

B10. A third order Runge-Kutta method for the ordinary differential equation

$$y' = f(x, y)$$

has the following recurrence relation

$$y_{n+1} = y_n + ak_1 + bk_2 + ck_3 \ ,$$

where

$$k_1 = hf(x_n, y_n) \ ,$$

$$k_2 = hf(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_1) \ ,$$

$$k_3 = hf(x_n + h, y_n + k_2) \ .$$

Determine the coefficients a, b, c so that the scheme is of third order. Show all your derivations.