APPLIED MATHEMATICS and STATISTICS
DOCTORAL QUALIFYING EXAMINATION
in COMPUTATIONAL APPLIED MATHEMATICS

Spring 2003

(CLOSED BOOK EXAM)

This is a two part exam.
In part A, solve 4 out of 5 problems for full credit.
In part B, you must also solve 4 out of 5 problems for full credit.

Indicate below which problems you have attempted by circling the appropriate numbers:

Part A:	1	2	3	4	5
Part B:	6	7	8	9	10

NAME ________________________________

Start each answer on its corresponding question page. Print your name, and the appropriate question number at the top of any extra pages used to answer any question. Hand in all answer pages.

Date of Exam: Fri., Jan. 24, 2003

Time: 12:00 – 4:00 PM

Place: SBS, N-105

A1. Consider the differential equation $y'' = y^2, \qquad y(0) = 0.$

a) We are seeking the solution in the series form $y = \sum_{n=0}^{\infty} a_n t^n$. Express the solution up to and including the t^4 term. Is there a unique solution? Support your answer.

b) Find a nontrivial solution, up to and including the t^4 term, which satisfies

$$\sum_{n=0}^{4} y^{(n)}(0) = 0,$$

where $y^{(n)}$ denotes the n-th derivative of y.

A2. Let f and g be two solutions of the homogeneous linear differential equation

$$u'' + p(x)u' + q(x)u = 0, \qquad \text{in } x \in [a, b],$$

where $p(x)$ is a continuous function. Define the Wronskian $W(f, g; x)$ by

$$W(f, g; x) = f(x)g'(x) - f'(x)g(x).$$

a) Derive a differential equation for $W(f, g; x)$.

b) Prove that $W(f, g; x)$ is either: identically positive; identically negative; or identically zero.

A3. Solve the initial value problem

$$\begin{aligned} u_{tt} - c^2 u_{xx} &= 2t, \\ u(x,0) &= x^2, \\ u_t(x,0) &= 1. \end{aligned}$$

A4. Compute the explicit solution to

$$\begin{aligned} xu_x + yu_y + u_z &= u^2, \\ u|_{z=0} &= h(x, y), \end{aligned}$$

where $h \in C^1(R^2)$.

A5. Suppose that f is an entire function and that there is a bounded sequence of distinct real numbers $a_1, a_2, \ldots, a_n, \ldots$ such that $f(a_k)$ is real for each k.

a) Prove that $f(x)$ is real for all real x.

b) Furthermore, if

$$a_1 > a_2 > \ldots > a_n > \ldots > 0,$$
$$\lim_{k \to \infty} a_k = 0,$$
$$f(a_{2n+1}) = f(a_{2n}),$$

for all n, prove that f must be a constant.

B6. Find a full QR factorization for the matrix

$$A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ \sqrt{2} & 2 \end{pmatrix}$$

by Householder triangularization. Show all your work.

B7. For a given matrix $A \in \mathbf{R}^{m \times m}$, Arnoldi iteration is defined by $AQ_n = Q_{n+1}\tilde{H}_n$, for $n = 1, 2, \ldots$, where Q_n is an $m \times n$ matrix of orthonormal, unit column vectors and $\tilde{H}_n$ is an $(n+1) \times n$ upper Hessenberg matrix. The $n \times n$ matrix H_n, which is the upper part of $\tilde{H}_n$, contains information on the eigenvalues of A.

a) Express Aq_n as a linear combination of $\{q_1, \ldots, q_{n+1}\}$ where q_j is the j-th column vector of Q_{n+1}

Answer questions b) and c) for the following matrix,

$$A = \begin{pmatrix} 0 & 3 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} .$$

b) Starting from $q_1 = (0, 1, 0)^t$, find Q_2 and $\tilde{H}_2$.

c) Confirm that the two eigenvalues of H_2 are also eigenvalues of A.

B8. Consider the set of points $\{(-1,0),(0,1),(1,0)\}$.

a) Find the quadratic polynomial interpolating these points. Give both the Lagrange and Newton forms of this polynomial.

b) Find the natural cubic spline interpolating these points.

c) Find the straight line that fits the data best in the least-squares sense.

B9. Consider the finite-difference approximation

$$f''(x) \approx \frac{f(x + \Delta x) - 2f(x) + f(x - \Delta x)}{\Delta x^2}.$$

a) Find the first nonzero term in the Taylor series expansion for the error. This term will have the form $C\Delta x^d$, where C and d are to be determined.

b) Use Richardson extrapolation to find a more accurate finite-difference approximation for $f''(x)$, with leading error term being two orders higher than the scheme in a).

B10. Consider the initial value problem

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix},$$

with initial values $x(0) = 1$ and $y(0) = 0$.

a) Find the analytic solution $x(t)$ and $y(t)$, and show that

$$x^2(t) + y^2(t) = 1$$

for all values of t.

b) Suppose that x_N and y_N are estimates of $x(\pi)$ and $y(\pi)$ computed using N steps of Euler's method. Show

$$x_N^2 + y_N^2 = 1 + \frac{\pi^2}{N} + O(N^{-2}).$$

c) Consider an alternative method given by

$$x_{n+1} = (1 - h^2)^{1/2} x_n + h y_n, \qquad y_{n+1} = (1 - h^2)^{1/2} y_n - h x_n,$$

where $h = \pi/N$ and $N \geq 4$. Show that

$$x_n^2 + y_n^2 = 1$$

for all n.

d) For the alternative method in c), show that

$$x(\pi) - x_N = -\frac{\pi^6}{72 N^4} + O(N^{-6}).$$