APPLIED MATHEMATICS and STATISTICS

DOCTORAL QUALIFYING EXAMINATION

in COMPUTATIONAL APPLIED MATHEMATICS

SPRING 2004

(CLOSED BOOK EXAM)

This is a two part exam.
In part A, solve 4 out of 5 problems for full credit.
In part B, you must also solve 4 out of 5 problems for full credit.

Indicate below which problems you have attempted by circling the appropriate numbers:

Part A:	1	2	3	4	5
Part B:	6	7	8	9	10

NAME ______________________________

Start each answer on its corresponding question page. Print your name, and the appropriate question number at the top of any extra pages used to answer any question. Hand in all answer pages.

Date of Exam: Wed., Jan. 28, 2004

Time: 12:00 – 4:00 PM

Place: Math Rm. 122-A

A1. Assume λ is real and consider

$$y'' + \lambda^2 y = 0,$$

subject to

$$y(0) + y'(0) = 0, \quad y(1) = 0.$$

a) Show that $\lambda_0 = 0$ is an eigenvalue.

b) If $\lambda > 0$, show that $\tan \lambda = \lambda$.

c) Find the eigenfunctions.

A2. Find the (real valued) general solution of

$$\frac{d^4y}{dt^4} + 4\frac{d^2y}{dt^2} = 16t^2.$$

A3. Solve the initial value problem:

$$u_{tt} - u_{xx} = 0,$$
$$u(x,0) = \begin{cases} \sin^2 \pi x & \text{if } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$
$$u_t(x,0) = 0$$

Find the solution at $t = 2$ for

a) $-\infty < x < \infty$.

b) $0 < x < \infty$ and $u(0,t) = 0$.

c) $0 < x < \infty$ and $u_x(0,t) = 0$.

d) $0 < x < 1$ and $u(0,t) = u(1,t) = 0$.

A4. Find the maximun value of u in the following boundary value problems:

a)

$$\Delta u = 0, \quad x^2 + y^2 < 1$$
$$u(x,y)|_{x^2+y^2=1} = x^2 y + y^3$$

b)

$$u_t = u_{xx}, \quad 0 < t < 1, \quad 0 < x < 1$$
$$u(0,t) = 2t^2 - t,$$
$$u(1,t) = \sin \pi t,$$
$$u(x,0) = x(1-x)$$

A5. Let $f(z)$ be an entire function such that $|f(z)| \leq M|z|^m$ for large $|z|$, where M is constant and $m \geq 0$ is an integer. Show that $f(z)$ is a polynomial of degree at most m.
Hint: Use Cauchy's inequality on derivatives of f to establish

$$|f^{(m+1)}(\alpha)| \leq \frac{M(m+1)!(R+|\alpha|)^m}{R^{m+1}}$$

and thus deduce $f^{(m+1)}(\alpha) = 0$.

B6. Consider the block matrix product

$$\begin{bmatrix} W & X \\ Y & Z \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} E & F \\ G & H \end{bmatrix},$$

where, for simplicity, all the matrices $A, B, \ldots, Y, Z$ are assumed to be square and of the same dimension. Strassen showed that W, X, Y, Z can also be computed by the formulas

$$\begin{aligned} P_1 &= (A + D)(E + H), & P_5 &= (A + B)H, \\ P_2 &= (C + D)E, & P_6 &= (C - A)(E + F), \\ P_3 &= A(F - H), & P_7 &= (B - D)(G + H), \\ P_4 &= D(G - H), & & \\ W &= P_1 + P_4 - P_5 + P_7, & Y &= P_2 + P_4, \\ X &= P_3 + P_5, & Z &= P_1 + P_3 - P_2 + P_6. \end{aligned}$$

a) Let $n(k)$ be the total number of scalar additions/subtractions and multiplications for multiplying matrices of dimension $m = 2^k$. Prove that

$$n(k) = 7n(k - 1) + 18 \cdot 2^{2k-2}$$

for $k > 0$ with $n(0) = 1$.

b) Find $n(k)$ explicitly.

c) Show that, by applying Strassen's formulas recursively, one can obtain an algorithm for multiplying matrices of dimension $m = 2^k$ with an operation count smaller than $O(m^3)$ as $m \to \infty$.

B7. Let $\mathbf{F}$ be the set of real numbers which can be stored on a computer memory where $\text{fl}(x) \in \mathbf{F}$ denotes the number corresponding to a real number $x \in \mathbf{R}$. For the arithmetic operation of addition, consider the floating point operation $\oplus$ which satisfies the following two axioms:

A1. For all $x \in \mathbf{R}$, there exists ϵ with $|\epsilon| \leq \epsilon_{\text{machine}}$ such that $\text{fl}(x) = x(1+\epsilon)$.

A2. For all $x, y \in \mathbf{F}$, there exists ϵ with $|\epsilon| \leq \epsilon_{\text{machine}}$ such that $x \oplus y = (x+y)(1+\epsilon)$

Show that, for any $x, y \in \mathbf{R}$, there exist $\tilde{x}, \tilde{y} \in \mathbf{R}$ where

$$\text{fl}(x) \oplus \text{fl}(y) = \tilde{x} + \tilde{y}$$

and $|x - \tilde{x}|/|x| = O(\epsilon_{\text{machine}})$ and $|y - \tilde{y}|/|y| = O(\epsilon_{\text{machine}})$. In other words, show $\oplus$ is a backward stable algorithm.

B8. Consider a function f which is twice continuously differentiable on the interval $I = [-1, 1]$. Interpolate the function by a linear polynomial through the support points

$$(x_i, f(x_i)), \quad i = 0, 1, \qquad x_0, x_1 \in I.$$

Verify that

$$\alpha = \frac{1}{2} \max_{\xi \in I} |f''(\xi)| \max_{x \in I} |(x - x_0)(x - x_1)|$$

is an upper bound for the maximal absolute interpolation error on the interval I. What values of x_0, x_1 minimize α?

B9. Let $f \in C^6[-1,1]$, and let $P(x)$ be the Hermite interpolation polynomial of degree 5 with

$$P(x_i) = f(x_i), \qquad P'(x_i) = f'(x_i), \qquad x_i = -1, 0, 1.$$

a) Show that

$$\int_{-1}^{1} P(t)dt = \frac{7}{15}f(-1) + \frac{16}{15}f(0) + \frac{7}{15}f(1) + \frac{1}{15}f'(-1) - \frac{1}{15}f'(1).$$

b) By construction, the above formula represents an integration rule which is exact for all polynomials of degree 5 or less. Show that it need not be exact for polynomials of degree 6.

c) Derive an error formula for the integration rule in (a).

d) Given a uniform partition

$$x_i = a + ih, \qquad i = 0, \cdots, 2n, \qquad h = (b-a)/2n$$

of the interval $[a, b]$, what composite integration rule can be based on the integration rule in (a)?

B10. Consider the single step method

$$u_{j+1} = u_j + hf(x_j + \frac{h}{2}, u_j + \frac{h}{2} f(x_j, u_j))$$

a) Show the method has second order accuracy if f is twice continuously differentiable.

b) Use the method to solve the initial value problem

$$u' + 100u = 100, \quad u(0) = 2.$$

c) Determine the stability condition on the step size h for the discrete solution in b).