APPLIED MATHEMATICS and STATISTICS
DOCTORAL QUALIFYING EXAMINATION
in COMPUTATIONAL APPLIED MATHEMATICS

Spring 2005 (January)

(CLOSED BOOK EXAM)

This is a two part exam.
In part A, solve 4 out of 5 problems for full credit.
In part B, you must also solve 4 out of 5 problems for full credit.

Indicate below which problems you have attempted by circling the appropriate numbers:

Part A:	1	2	3	4	5
Part B:	6	7	8	9	10

NAME ________________________________

Start each answer on its corresponding question page. Print your name, and the appropriate question number at the top of any extra pages used to answer any question. Hand in all answer pages.

Date of Exam: Wed., Jan. 26, 2005

Time: 9:00 – 1:00 PM

Place: SBS Rm. N-110

A1. Suppose that $\phi_1(t)$ and $\phi_2(t)$ are a solutions of a second order linear homogenous ODE that has an ordinary point at $t = 0$, and that $\phi_1(0) = \phi_2(0) = 0$. Show that $\phi_1(t)$ and $\phi_2(t)$ are linearly dependent.

A2. Find a fundamental set of real-valued solutions of

$$t^2y'' + 5ty' + 5y = 0$$

on $(0, \infty)$.

A3. Solve $u_x^2 + yu_y - u = 0$ with the initial condition $u(x, 1) = x^2/4 + 1$.

A4.

a) Analyze the dispersion relation for the linearized KdV equation

$$u_t + cu_x + \gamma^2 u_{xxx} = 0.$$

b) By considering the solution to the linear advection equation

$$u_t + cu_x = 0, \quad u(x, 0) = sin(x) + sin(3x),$$

discuss the qualitative picture of the solution to the linearized KdV equation, with the same initial condition, that emerges from analysis of the dispersion relation as the value of γ is increased from 0.

A5. Using an appropriate contour integral, evaluate the following:

$$\int_0^{2\pi} \frac{2 + \cos x}{4 + \cos x}\, dx.$$

B6. Consider the following algorithm to solve the least squares problem to minimize $\|b - Ax\|_2$.

Least Squares via QR Factorization

1. Compute the reduced QR factorization $A = \hat{Q}\hat{R}$.
2. Compute the vector $\hat{Q}^* b$.
3. Solve the upper-triangular system $\hat{R}x = \hat{Q}^* b$ for x.

Explain how to compute the square of the norm of minimum residual $\|b - Ax\|_2^2$ while running this algorithm. The count of multiplications, additions and subtractions should not exceed $2m + 2n - 1$ for a matrix of dimension $m \times n$ $(m > n)$.

Hint: $Ax = Pb$ where P is the orthogonal projector $\hat{Q}\hat{Q}^*$.

B7. Let

$$A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3/2 & 1/2 \\ 0 & 1/2 & 3/2 \end{pmatrix}.$$

This matrix is obtained as the product $X\Lambda X^{-1}$ where

$$X = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{pmatrix} \quad \text{and} \quad \Lambda = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

a) By an explicit calculation, prove that the power iteration starting from the vector $(1, 1, -1)^T$ produces a sequence of vectors converging to the eigenvector corresponding to the largest eigenvalue.

b) Arnoldi iteration gives us Q_n and $\tilde{H}_n$ related by $AQ_n = Q_{n+1}\tilde{H}_n$ for a general square matrix A of dimension $m \times m$ where Q_n is an $m \times n$ matrix with orthonormal columns $q_i, i = 1, \ldots, n$ and $\tilde{H}_n = (h_{ij})$ is an $(n+1) \times n$ upper Hessenberg matrix. Write the $(n+1)$-term recurrence relation involving the vectors $q_1, \ldots, q_{n+1}$.

c) Apply the Arnoldi iteration algorithm to matrix A given above, starting from $b = (0, 0, 1)^T$ and continuing until you find two eigenvalues of the matrix A.

B8.

a) Consider the problem of polynomial interpolation of the function $y(x)$ over the interval $[x_1, x_2]$. Assume the values $y(x_1)$, $y(x_2)$, $y'(x_1)$ and $y'(x_2)$ are known (where y' denotes dy/dx).

i) Obtain an interpolating polynomial for $y(x)$ on $[x_1, x_2]$ which uses these four known values.

ii) Give the functional form of the interpolation error for this polynomial valid for any $\bar{x} \in [1, 2]$.

b) If the differential equation

$$y' = f(y)$$

is integrated over the interval $[x_1, x_2]$ and Simpson's rule is used to evaluate the RHS integral, we get

$$y_2 - y_1 - \frac{h}{6}\left[f(y_1) + 4f(y_{1.5}) + f(y_2)\right] + \text{ Error } = 0, \qquad (1)$$

where $h \equiv x_2 - x_1$, $y_i \equiv y(x_i), i = 1, 2$, and $y_{1.5} \equiv y((x_1 + x_2)/2)$. If an interpolated value for $y_{1.5}$, obtained via the formula derived in part a), is used in equation (1), show that the order of the "Error" term remains unchanged. What assumption on the function $f(y)$ is made in order to show this?

B9.

a) One way to solve the quadratic $f(x) \equiv x^2 - x - 2 = 0$ is by the point iterative method

$$x_{k+1} = x_k - \frac{f(x_k)}{m}, \qquad m \neq 0.$$

Show that it is possible to choose a value of m that will make the convergence of this scheme quadratic for one of the roots, but not for the other. Find the appropriate value of m needed for each root.

b) Apply Newton's method to the function $f(x) = 1/x - a$ to find $g(x)$ such that the iterates

$$x_{k+1} = g(x_k)$$

converge to $1/a$. Show that this iteration formula can be written in the interesting form

$$x_{k+1} f(x_{k+1}) = (x_k f(x_k))^2.$$

B10. Consider the sequence $\phi_0(x), \phi_1(x), \ldots, \phi_{n+1}(x)$ of orthogonal polynomials for a weight function $w(x)$ on $[-1,1]$; i.e.

$$\langle \phi_i, \phi_j \rangle \equiv \int_{-1}^{1} \phi_i(x)\phi_j(x)w(x)dx = 0 \quad \text{for } i \neq j.$$

Let $\xi_0, \ldots, \xi_n$ denote the zeros of ϕ_{n+1}.
Let $p_{n+1}(x)$ be a polynomial of degree $\leq n+1$.
Let $q_n(x)$ be the polynomial of degree $\leq n$ which **interpolates** $p_{n+1}(x)$ at the points $\xi_0, \ldots, \xi_n$.
Let $r_n(x)$ be the polynomial of degree $\leq n$ which is the **best approximating polynomial** to p_{n+1} with respect to the norm

$$\|g\|^2 \equiv \langle g, g \rangle = \int_{-1}^{1} g(x)^2 w(x)dx.$$

Derive the relationship between $q_n(x)$ and $r_n(x)$.