APPLIED MATHEMATICS and STATISTICS

DOCTORAL QUALIFYING EXAMINATION

in COMPUTATIONAL APPLIED MATHEMATICS

Spring 2006 (June)

(CLOSED BOOK EXAM)

This is a two part exam.
In part A, solve 4 out of 5 problems for full credit.
In part B, you must also solve 4 out of 5 problems for full credit.

Indicate below which problems you have attempted by circling the appropriate numbers:

Part A:	1	2	3	4	5
Part B:	6	7	8	9	10

NAME ____

Start each answer on its corresponding question page. Print your name, and the appropriate question number at the top of any extra pages used to answer any question. Hand in all answer pages.

Date of Exam: Wed., June 14, 2006

Time: 9:00 AM – 1:00 PM

Place: Grad Chemistry Rm. 126

A1. Find all λ such that the boundary value problem

$$y'' + \lambda y = 0, \quad y'(0) = y'(L) = 0$$

has nontrivial solution on the interval $[0, L]$, $L > 0$. For each λ indicate the corresponding solution.

A2. Let $I = (a, b)$ be an interval and p, q be continuous real functions on I such that $p(t) > q(t)$ for $t \in I$. On I, let f and g be real and nontrivial solutions of $y'' + p(t)y = 0$ and $y'' + q(t)y = 0$, respectively. Prove that if t_1 and t_2, with $a < t_1 < t_2 < b$, are successive zeros of g, then $f(\tau) = 0$ for some $\tau \in (t_1, t_2)$.

A3. Derive the (formal) solution to the initial/boundary value problem

$$
\begin{aligned}
u_{tt} - \nabla^2 u = A(x)\sin\omega t, &\quad \text{for } x \in \Omega, \quad t > 0 \\
u(x,0) = 0 &\quad \text{for } x \in \Omega \\
u_t(x,0) = 0 &\quad \text{for } x \in \Omega \\
u(x,t) = 0 &\quad \text{for } x \in \delta\Omega, \quad t > 0
\end{aligned}
$$

for $\Omega \in R^n$ in terms of the eigenfunctions $\phi_n(x)$ of the Laplacian on Ω. (*Hint*: Use Duhamel's principle.) List the specific properties of the eigenfunctions/eigenvalues that are used in this solution technique.

A4. Solve

$$y^2 u_{xx} - 2y u_{xy} + u_{yy} = u_x + 6y$$

using characteristics and reduction to canonical form.

A5.

a) Determine the radius of the circle of convergence of the power series

$$f(z) = \sum_{k=0}^{\infty} z^{k!}.$$

Show that the circle of convergence is a natural boundary beyond which the sum -function $f(z)$ has no analytic continuation.

b) Using the calculus of residues and an appropriate contour integral, evaluate

$$\int_0^{\infty} \frac{\sin^2 x}{x^2} dx.$$

B6. If A is any nonsingular matrix and its condition number is $\kappa(A) = \parallel A \parallel \parallel A^{-1} \parallel$, prove that if $A + E$ is singular, then

$$\frac{\parallel E \parallel}{\parallel A \parallel} \geq \frac{1}{\kappa(A)} \quad \text{or} \quad \kappa(A) \geq \frac{\parallel A \parallel}{\parallel E \parallel}$$

Use this result to find a lower bound (the best you can) for $\kappa_\infty(A) = \parallel A \parallel_\infty \parallel A^{-1} \parallel_\infty$ when A is the following (ill-conditioned) nonsingular matrix

$$A = \begin{bmatrix} 1 & 0.0002 & 0.9999 \\ 2 & -1 & 1.0002 \\ 2.9997 & 1.00001 & 4 \end{bmatrix}$$

That is, find a suitable perturbation of A so that $A + E$ is singular and $\parallel A \parallel / \parallel E \parallel$ is large.

B7. Let A be an arbitrary 3×3 matrix. For each of the following, can A be reduced to the given zero(0)/nonzero(X) pattern by multiplication on the left by plane rotation matrices. Justify your answer.

$$\text{(a)} \quad \begin{bmatrix} X & X & X \\ 0 & 0 & X \\ 0 & 0 & X \end{bmatrix} \qquad \text{(b)} \quad \begin{bmatrix} X & X & 0 \\ X & 0 & X \\ 0 & X & X \end{bmatrix}$$

B8. Perform the LU decomposition of the matrix A, that is, find the lower and upper triangular matrices L and U such that $A = LU$.

$$A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 4 & 3 & 3 & 1 \\ 8 & 7 & 9 & 5 \\ 6 & 7 & 9 & 8 \end{bmatrix}$$

B9.

a) Determine the orthogonal polynomials $\phi_n(x)$, $\quad n = 0, 1, 2, 3$ with leading coefficient 1, for the weight function $w(x) = 1 + x^2$, $\quad -1 \leq x \leq 1$.

b) Calculate the best, least squares fitting polynomial of degree ≤ 3 to e^x with respect to the weight function $w(x) = 1 + x^2$ over the interval $-1 \leq x \leq 1$.

B10. Consider the following 1-parameter family of one-step methods

$$u_{n+1} = u_n + h(1 - \frac{\alpha}{2})f(x_n, u_n) + h\frac{\alpha}{2}f(x_{n+1}, u_{n+1}).$$

a) Study the consistency of this scheme as a function of α.

b) For $0 < \alpha < 1$, determine the values of h for which the method is absolutely stable when applied to the ODE

$$y'(x) = -10y(x), \quad x > 0,$$
$$y(0) = 1.$$