

**APPLIED MATHEMATICS and STATISTICS
DOCTORAL QUALIFYING EXAMINATION
in COMPUTATIONAL APPLIED MATHEMATICS**

Spring 2007 (January)

(CLOSED BOOK EXAM)

This is a two part exam.

In part A, solve 4 out of 5 problems for full credit.

In part B, solve 4 out of 5 problems for full credit.

Indicate below which problems you have attempted by circling the appropriate numbers:

Part A:	1	2	3	4	5
Part B:	6	7	8	9	10

NAME _____

Start each answer on its corresponding question page. Print your name, and the appropriate question number at the top of any extra pages used to answer any question. Hand in all answer pages.

Date of Exam: Wed., Jan. 24, 2007

Time: 9:00 – 1:00 PM

Place: TBA

A1. Find a positive-valued solution to the equation

$$\frac{dy}{dx} + xy = \frac{x}{y^3},$$

satisfying $y(0) = 2^{1/4}$.

A2. Find the solution to the initial value problem

$$y'' + 2y' + y = 1, \quad y(0) = 2, \quad y'(0) = -2.$$

A3. Solve the Laplace Equation for the unbounded strip

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < \pi, \quad y > 0$$

satisfying the boundary conditions

$$u(0, y) = 0, \quad u(\pi, y) = 0, \quad u(x, 0) = (1 - x^2) \cos x$$

and the regularity condition $u(x, y) \rightarrow 0$, as $x^2 + y^2 \rightarrow \infty$.

A4. a) Illustrate the use of the Fourier or Laplace transforms by solving the one-dimensional heat equation,

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{\kappa} \frac{\partial u}{\partial t},$$

on the domain $-\infty < x < \infty$, $t > 0$ with the initial condition $u(x, 0) = f(x)$.

b) Show that the solution for the backward heat equation

$$\frac{\partial^2 u}{\partial x^2} = -\frac{1}{\kappa} \frac{\partial u}{\partial t}$$

is unstable; i.e. it does not vary continuously with the initial data.

Hint: Consider an appropriate variable-separable solution.

A5. Let $f(z)$ be an analytic function of the complex variable z on a domain D . Let C be a smooth closed curve inside D and suppose that C and its interior E are mapped on to the unit disc $|w| \leq 1$. Prove that the points on the boundary $|w| = 1$ can not be the image of an interior point of E .

B6. a) Prove that matrices A and Q^*AQ have the same eigenvalues.

b) Prove that if A is an $m \times m$ Hermitian matrix, then A can be reduced to tridiagonal form through a similarity transformation, $T = Q^*AQ$. Count the number of operations needed for the reduction.

c) In implementing the power method to find the eigenvalue closest to a number μ , one can use:

(i) direct iteration, by solving $(A - \mu I)x^{k+1} = x^k$;

or

(ii) first reduce the matrix to T , and then solve $(T - \mu I)x^{k+1} = x^k$, through the double sweep (Thomas) method.

If convergence for both methods requires 20 iterations, count the number of operations that would be needed to implement both (i) and (ii).

B7. a) Prove that if δA satisfies

$$\|\delta A\| < \frac{1}{\|A^{-1}\|}$$

in any matrix norm, then $A + \delta A$ is nonsingular.

Hint: If $A + \delta A$ is singular then $(A + \delta A)x = 0$ has nontrivial solutions.

b) Use a) to show that the condition number,

$$\kappa \leq \|A\| \|A^{-1}\|,$$

of a nonsingular matrix A can also be defined as

$$\frac{1}{\kappa} \geq \min \left\{ \frac{\|\delta A\|}{\|A\|} : A + \delta A \text{ is singular} \right\}.$$

- B8.** a) If P is an $m \times m$ projector, show that $\|P\|_2 \geq 1$. What will make $\|P\|_2 = 1$?
- b) If P is an orthogonal projector, show that $\|Px\|_2 \leq \|x\|_2$. Give a geometrical explanation for this.
- c) If P is an orthogonal projector, show that $I - 2P$ is unitary.
- d) Show how to orthogonally project a vector $x = (2 \ 3 \ 2 \ 1 \ 2 \ 2)^T$ onto the subspace which is orthogonal to $x - v$, where $v = (0 \ 0 \ 0 \ \|x\|_2 \ 0 \ 0)^T$.
- e) Find a unitary matrix F such that $Fx = (0 \ 0 \ 0 \ \|x\|_2 \ 0 \ 0)^T$.
- f) Give a geometrical explanation for e).

B9. Using the lowest order interpolating polynomial possible, design a centered finite difference scheme for $f'''(a)$. Assume evenly spaced interpolating points. The difference scheme should only require evaluations of $f()$, and not require evaluation of derivatives of $f()$. Provide an analysis for the order of error of the scheme.

Note: For credit, your scheme must be based upon an interpolating polynomial.

B10. a) Determine the orthogonal polynomials $\phi_n(x)$, $n = 0, 1, 2, 3$ with leading coefficient 1, for the weight function $w(x) = 1 + x^2$ on the interval $-1 \leq x \leq 1$.

b) Calculate the best, least squares fitting polynomial of degree ≤ 3 to e^x with respect to the weight function $w(x) = 1 + x^2$ over this interval.