

**APPLIED MATHEMATICS and STATISTICS**  
**DOCTORAL QUALIFYING EXAMINATION**  
**in NUMERICAL ANALYSIS and SCIENTIFIC COMPUTATION**

**Spring 1998**

**(CLOSED BOOK EXAM)**

Solve any six (6) problems for full credit. Indicate below which problems you have attempted by circling the appropriate numbers:

1            2            3            4            5            6            7            8

**NAME** \_\_\_\_\_

Start each answer on its corresponding question page. Attach all extra sheets you use for an answer to the appropriate question page. Hand in all answer pages. Print your name at the top of each page handed in.

Date of Exam: Wed, Feb. 18, 1998

Time: Noon – 3:00 PM

Place: Math Tower, 1-122A

1. (a) Show that the iterative scheme

$$x_{k+1} = g(x_k), \quad k = 0, 1, 2, \dots$$

converges if

$$\|g'(x)\| \leq \lambda < 1,$$

where  $x, g \in R^n$  and  $g'(x)$  is the Jacobian of  $g$ .

(b) Prove that Newton's method for a single nonlinear equation has quadratic convergence.

**2.** Describe an efficient computational method for finding the minimum norm least squares solution of a system of linear equations  $Ax = b$ , where  $A$  is an  $m \times n$  matrix of rank  $r$ ,  $r < \min(m, n)$ ,  $x$  and  $b$  are, respectively, vectors of dimension  $n$  and  $m$ .

**3.** Describe in detail an efficient method for computing the minimum eigenvalue of a symmetric positive definite matrix. Give a concise and clear derivation of the algorithm

4. Let  $x_0, \dots, x_{n-1}$  be  $n$  distinct points in  $\mathcal{R}$ . Let  $x_n$  be an *arbitrary* point in  $\mathcal{R}$ . Consider the problem of constructing a polynomial  $P_n(x)$  of degree  $\leq n$  which agrees with  $y_i \equiv f(x_i)$  at  $x_0, \dots, x_{n-1}$  and agrees with  $y'_n \equiv f'(x_n)$  at  $x_n$ .

- (a) For what points  $x_n \in \mathcal{R}$  can such a  $P_n(x)$  **not** be constructed. Give the number of such points and information on their location.
- (b) Otherwise, show that a unique interpolating polynomial  $P_n(x)$  can be constructed.

5. (a) Determine the polynomials  $\phi_k(x)$ ,  $k = 0, 1, 2, 3$  each having leading (i.e. highest order) coefficient 1, that are orthogonal on the interval  $-1 \leq x \leq 1$  with respect to the weight function  $w(x) = 1 + x^2$ .
- (b) Given data  $f(x_i)$  for  $n$  distinct  $x_i$  uniformly covering the interval  $[-1, 1]$ , consider the problem of finding the functional form

$$F(x) = \sum_{k=0}^3 c_k \phi_k(x)$$

which minimizes the error

$$d \equiv \left( \sum_{i=1}^n (1 + x_i^2) (f(x_i) - F(x_i))^2 \right)^{1/2}.$$

Give the set of linear equations that must be solved to determine the values of the  $c_k$ . Assuming that the number of points  $n$  is very large, find an approximate, simplified integral expression for each  $c_k$ .

6. Consider the ODE

$$\frac{dy}{dx} = f(x, y(x)) .$$

Define

$$f_i \equiv f(x_i, y_i), \quad y_i \equiv y(x_i) .$$

(a) Derive the local error formula for the numerical scheme

$$y_{n+1} = y_n + \frac{h}{24}(9f_{n+1} + 19f_n - 5f_{n-1} + f_{n-2}) .$$

(b) Derive the conditions under which the iteration

$$y_{n+1}^{(k+1)} = y_n + \frac{h}{24}(9f_{n+1}^{(k)} + 19f_n - 5f_{n-1} + f_{n-2})$$

converges, where

$$f_n^k \equiv f(x_{n+1}, y_{n+1}^{(k)}) .$$

7. Consider the following finite difference scheme

$$\frac{v_{m+1}^{n+1} - 2v_m^n + v_{m-1}^{n+1}}{2k} + a \frac{v_{m+1}^{n+1} - v_{m-1}^{n+1}}{2h} = f_m$$

for solving the differential equation  $u_t + au_x = f$ , where  $v_m^n$  is the finite difference approximation to  $u(nk, mh)$ .

- (a) What are the conditions on  $a, k$ , and  $h$  for this scheme to be stable?
- (b) Analyze the order of accuracy of this scheme.
- (c) If the scheme is modified by adding a term  $bv_m^n$  to the left hand side, so that it becomes a solution scheme for the equation  $u_t + au_x + bu = f$ , how does that change the accuracy and stability results?

8. Consider the coupled equations

$$\frac{du}{dx} + a \frac{dv}{dy} + \lambda u = 0$$
$$\frac{dv}{dx} + b \frac{du}{dy} + \mu v = 0$$

- (a) This system may be elliptic, parabolic, or hyperbolic. Specify the conditions on the parameters  $a, b, \lambda, \mu$  which give rise to each type.
- (b) For the elliptic case, write down a five point finite difference scheme to solve for  $u$ .