APPLIED MATHEMATICS and STATISTICS

DOCTORAL QUALIFYING EXAMINATION

in NUMERICAL ANALYSIS and SCIENTIFIC COMPUTATION

Spring 1999

(CLOSED BOOK EXAM)

Solve any six (6) problems for full credit. Indicate below which problems you have attempted by circling the appropriate numbers:

1 2 3 4 5 6 7 8

NAME ________________________________

Start each answer on its corresponding question page. Attach all extra sheets you use for an answer to the appropriate question page. Hand in all answer pages. Print your name at the top of each page handed in.

Date of Exam: Fri., Jan. 22, 1999

Time: 9:00 – 12:00 Noon

Place: Math Tower, 1-122A

1. A is an m by n matrix of rank r, where $r < m < n$. Define the singular value decomposition of A. Show that the singular values are the positive square roots of the non-zero eigenvalues of $A^T A$ and also of AA^T. Use the singular value decomposition to prove that

$$\lim_{\epsilon \to 0} (A^T A + \epsilon I)^{-1} A^T = A^+$$

where A^+ is the Moore-Penrose generalized inverse of A.

2. Show that δx obtained from

$$\left[(f')^T f' + \epsilon I\right] \delta x = -(f')^T f$$

minimizes the functional

$$\|f' \delta x + f\|_2^2 + \epsilon \|\delta x\|_2^2$$

where f' is the Jacobian of f at x, $f(x^*) = 0$, $x^* = x + \delta x$, and $x, f \in R^n$.

3. Describe the Gauss-Seidel method for the iterative solution of the system of linear equations $Ax = b$.

Derive the SOR (successive over-relaxation) formula using the Gauss-Seidel formula.

Prove that SOR converges if $0 < \omega < 2$, where ω is the relaxation parameter.

4.

(a) Let $g(x) \equiv f[x_0, \ldots, x_k, x]$. Prove that

$$g[y_0, \ldots, y_n] = f[x_0, \ldots, x_k, y_0, \ldots, y_n]$$

where $h[\ldots]$ denotes the divided difference for any function $h(x)$.

(b) Prove

$$\frac{d^n g(x)}{dx^n} = n! f[x_0, \ldots, x_k, x, \ldots, x]$$

where x appears $n + 1$ times on the right hand side.

5. Calculate the polynomial of degree ≤ 3 which minimizes

$$\int_0^\infty [e^{x/4} - p(x)]^2 e^{-x} dx$$

over all polynomials $p(x)$ of degree ≤ 3.
Hint: recall the three term relation for orthogonal polynomials

$$P_{i+1}(x) = A_i(x - B_i)P_i(x) - C_i P_{i-1}(x)$$

With

$$A_i = \frac{\alpha_{i+1}}{\alpha_i}, \qquad S_i = \langle P_i, P_i \rangle$$

$$B_i = \frac{\langle xP_i, P_i \rangle}{S_i}, \qquad C_i = \frac{A_i S_i}{A_{i-1} S_{i-1}},$$

and α_i is the coefficient of x^i in $P_i(x)$.

6. Consider the integral $\int_a^b f(x)dx$.

(a) Divide the interval $[a, b]$ into N uniform subintervals and apply the Trapezoid rule to each subinterval. Write down the numerical integration scheme $I(N)$ this gives.

(b) Suppose we have two approximate solutions $I(N)$ and $I(N/3)$ which are obtained using the above rule. What is the order of accuracy of $I(N)$? Of $I(N/3)$?

(c) Construct an approximation of higher order than $I(N)$ and $I(N/3)$ using only the values $I(N)$ and $I(N/3)$.

(d) What is the order of accuracy of this higher order approximation?

7. For the differential equation $y'(t) = f(t, y(t))$, $y(0) = y_0$, consider the integration scheme given by

$$y_{n+3} + 2y_{n+2} - y_{n+1} - 2y_n = 6hf(t_n, y_n)$$

where $t_n = nh$.

(a) Find the local error of this integration scheme.

(b) Determine the stability of this scheme.

8. Find the accuracy of the finite difference approximation

$$\frac{1}{6}v_{m+1}^{n+1} + \frac{2}{3}v_m^{n+1} + \frac{1}{6}v_{m-1}^{n+1} + \frac{a\lambda}{4}\left(v_{m+1}^{n+1} - v_{m-1}^{n+1}\right)$$
$$= \frac{1}{6}v_{m+1}^{n} + \frac{2}{3}v_m^{n} + \frac{1}{6}v_{m-1}^{n} - \frac{a\lambda}{4}\left(v_{m+1}^{n} - v_{m-1}^{n}\right)$$
$$+\frac{k}{12}\left(f_{m+1}^{n+1} + 4f_m^{n+1} + f_{m-1}^{n+1} + f_{m+1}^{n} + 4f_m^{n} + f_{m-1}^{n}\right)$$

to the PDE $u_t + au_x = f$. Here, $\lambda \equiv k/h$, where $t_n = nk$, $x_m = mh$.

9. Consider the finite difference approximation

$$\frac{1}{6}v_{m+1}^{n+1} + \frac{2}{3}v_m^{n+1} + \frac{1}{6}v_{m-1}^{n+1} + \frac{a\lambda}{4}\left(v_{m+1}^{n+1} - v_{m-1}^{n+1}\right)$$
$$= \frac{1}{6}v_{m+1}^{n} + \frac{2}{3}v_m^{n} + \frac{1}{6}v_{m-1}^{n} - \frac{a\lambda}{4}\left(v_{m+1}^{n} - v_{m-1}^{n}\right)$$

to the PDE $u_t + au_x = 0$. Here, $\lambda \equiv k/h$, where $t = nk$, $x = mh$.

(a) Characterize the stabilty of this scheme.

(b) Is the scheme dissipative? If so, of what order?

(c) Find the dispersion relation for this scheme. Is the scheme dispersive?

(d) Find the dispersion relation for the PDE. Is the PDE dispersive? Is the PDE dissipative?