Plan for Semester End

Choose topics from Chapters 11, 12, 13
Basic Options Theory
Lognormal stock price model
Black-Scholes Equation
Project 2 due Monday 12/14/2009 at 7:45PM
Basic Options Theory

Option: The right but not the obligation to buy (or sell) an asset under specified terms

Call: An option that gives the right to buy (benefits if price of the asset rises)

Put: An option that gives the right to sell (benefits if the price of the asset falls)

Premium: Price of an option

Exercise: When the option holder buys or sells
Buying and Writing Options

Writer of an option earns premium and has the obligation to deliver or buy an asset at the strike price. Hope nothing happens.

Writing options can expose the writer to risk

Writing a put: Obligates writer to purchase at a price that may be lower than the trading price

Writing a call: Obligates writer to sell at a price that may be higher than the trading price
Basic Options Theory

An option is a derivative security whose underlying asset is that one that can be bought or sold.

The value of an option depends on the value of the price of the underlying asset.

Example: How much is the Dec $130 call for IBM? How much is the Dec $120 put for IBM?
Basic Option Concepts

Underlying Asset: What can be bought/sold for a call/put.

Exercise (Strike) Price: Price at which call/put allows the asset to be purchased/sold.

Expiration: Defines the end of the time period during which the option is valid.

American Options: Exercised during valid period.

European: Exercised only at expiration.
Basic Option Concepts

Premium: Price for the option

Buyer of an option is “long” the option and the seller has “written” the option (or is short)

Example: 1 Dec 2009 $130 IBM European call is the right to purchase 100 shares of IBM at $130. Option expires on 12/19/2009. Option may be exercised only at expiry.
Basic Options Concepts

Most options are exchange traded on the Chicago Board of Options Exchange (CBOE)

Ticker: Each strike has a unique ticker

Strike: Moves in increments that depend on the price of a stock

Expiry: Stock options expire on third Friday of every month

Price: 1 option = 100 shares of stock
Value of Options at Expiry

Suppose you own a call option with strike \( K \). On the date of expiry the asset has price \( S \). What is the value, \( C \), of this option?

\[
C = \max (0, S - K)
\]

What is the value if the option is a put, \( P \)?

\[
P = \max (0, K - S)
\]

Call payoff is unbounded, put payoff is bounded.
Value of Options

Calls:
- In the Money \( S > K \)
- At the Money \( S = K \)
- Out of the Money \( S < K \)

Puts:
- In the Money \( S < K \)
- At the Money \( S = K \)
- Out of the Money \( S > K \)
IBM Dec Options

What is the payoff function for $130$ calls and puts?
Are the options in the money or out of the money?
What Effects Value of Options

Time: All things being equal, options with more time to expiry have higher premiums

Volatility: Stocks with higher volatility will have higher option premiums

Interest Rate: Purchasing the option is like purchasing the stock with very little money

Changes in the price of the underlying asset
How Much Is An Option Worth?

Answered in two different but equivalent ways by Fisher Black and Myron Scholes (1973) and Robert Merton (1973)

With a method for valuing derivatives, the market for options grew rapidly

Options give organizations the ability to transfer large amounts of risk quickly and efficiently
Simple Example

A stock is currently trading for $100
Over a given period the probabilities of the stock going to $120 and $80 are 75% and 25%
What should a $120 call option be worth?
Black-Scholes argued that the price of the option should be the price at which no arbitrage is possible
Arbitrage: The ability to gain profit with no risk
A Simple Example

Assume no transaction costs and that a bank will lend you money at 0% interest.
Assume you can buy fractional shares of stock.
Compute the expected value of your profit on one share of stock in one period (it is $15).
Should this be the value of the option?
This value provides arbitrage opportunities.
Black-Scholes-Merton

Black and Scholes used the “equilibrium” argument

Merton used a “synthetic” option to prove his point – the cost of a synthetic option should be the same as the real option

Using the previous example, price the option by observing the cost of a synthetic option

Merton calls the “Law of One Price”
Put-Call Parity

For European options, the combination of a put, a call and a risk-free loan has a payoff identical to the stock itself.

Put-Call Parity: Let $C$ and $P$ be the prices of a European call and a European put, both with strike price $K$ and both defined on the same stock with price $S$ and let $dK$ be a risk-free loan in the amount $K$.

$$C - P + dK = S$$
Put-Call Parity Example

IBM options (IBM @ $126.96)
Strike $125
Call: $3.80
Put: $1.80
Risk-Free Loan Amount: $125
Why Do We Trade Options?

Options allow us to change the payoff patterns of assets
Buying or shorting stock has a payoff pattern over time
Calls and puts on the stock have payoff patterns that are different
Protective Put

Puts a floor on how much money you can lose from owning a stock

Combination: One share of stock purchased for $S_0$, one put purchased for $P$ with exercise price $X$

Example: Buy 100 IBM on Friday @ $125.70
If the stock goes below $115, do not want to lose any more money
Buy one Jan $115 put for $0.96 ($9.60)
Butterfly Spread

Profitable if a stock price doesn’t move much

Constructed by buying two calls, with strike prices $K_1$ and $K_3$ and by selling two calls, both with strike price $K_2$, with $K_2$ near the current stock price and $K_1 < K_2 < K_3$

The Butterfly Spread is profitable if the stock price at expiration is at or close to $K_2$, otherwise the loss is minimal
IBM Butterfly Spread

IBM closed on Friday at $126.96

Choose $K_2$ first: $K_2 = $125 ($3.80)

$K_1 = $120 ($7.70)$ and $K_3 = $130 ($1.30)$

How much does one unit of this trade cost?

What does the payoff curve look like?

How much money would you make/lose if IBM is at $125 at expiry?
Single Period Binomial Options Price

One period

$S$: Initial price of a stock

At the end of the period the price is $uS$ with probability $p$ or $dS$ with probability $(1-p)$

At every period we may borrow/lend at a common risk-free rate $r$ where $R = 1 + r$ and $u > R > d$

No arbitrage opportunities
Single Period Binomial Options Price

Suppose you have a call option on this stock with strike price $K$ and expiry at end of period. Look at the trajectory of possible outcomes. Let each outcome for the call can be duplicated by $x$ dollar worth of stock and $b$ dollars worth of the risk-free asset. Use the no arbitrage principle to obtain a price for the call.
Pricing the Call Option

Let $C$ be the price of the call option.

What is the payoff structure of the call option for each outcome of the stock price?

Duplicate option payoff by purchasing of $x$ dollars of stock and borrowing $b$ dollars.

Solve for $x$ and $b$.

What do we know about $x$ and $b$?
Example of Binomial Pricing

We set 1 day as the period
The price of a stock is $50 today
Tomorrow the stock may go up by 10% or go down by 3%
The 1 day interest rate is 6%
The call option matures in 1 day and has an exercise price $K = $50
Price this option
A Multiplicative Model for Stock Prices

The multiplicative model for the future price of a stock has the form

\[ S(k + 1) = u(k)S(k) \quad \text{for} \quad k = 0,1,\ldots, N - 1 \]

\( u(k) \) represents the relative change of the stock price.

\( S(k) \) is the current price.

\( u(k) \) is independent of units.
The Lognormal Distribution

If we take the natural logarithm of both sides

$$\ln S(k + 1) = \ln S(k) + \ln u(k) \text{ for } k = 0, 1, 2, \ldots, N - 1$$

We can see that $u(k)$ is what moves the prices

Let $w(k) = \ln u(k)$ and let the $w(k)$’s be normally distributed, independent r.v.’s with mean $\nu$ and variance $\sigma^2$

Then the $w(k)$’s are lognormally distributed
Lognormal Prices

A recursive formula for the multiplicative model

\[ S(k) = u(k - 1)S(k - 1) \]
\[ = u(k - 1)u(k - 2)...u(0)S(0) \]

If we take the natural log of both sides

\[ \ln S(k) = \ln u(k - 1) + \ln u(k - 2) + ... + \ln S(0) \]
\[ = \ln S(0) + \sum_{i=0}^{k-1} \ln u(i) = \ln S(0) + \sum_{i=0}^{k-1} w(i) \]
Lognormal Prices

$lnS(0)$ is a constant
All the $w(i)$’s are normally distributed
By the Central Limit Theorem the sum of normal r.v.’s is again a normal r.v.
We are summing $k$ r.v.’s so the mean of the sum is $k\nu$, the variance is $k\sigma^2$
Under the multiplicative model prices are lognormal – how does this fit with reality?
This is a Very Cool Thing!

Histogram of IBM prices

October 19, 1987
Black Monday
But It Has Big Problems!

Fat Tails – On 10/19/1987 IBM lost 20%
If IBM prices were really lognormally distributed, this would never happen
The proverbial 10 S.D. event
Center Peakedness – The density under the observed distribution is smaller around mean
Lognormal assumptions underestimates the bad things and overestimates the good things!
Lognormal Stock Prices and Options

Suppose we go from our discrete multiplicative model to a continuous form

\[ \ln S(k+1) - \ln S(k) = w(k) \text{ where } w(k) = \ln u(k) \]

\( w(k) \)'s are identically distributed, uncorrelated normal random variables

Let’s say that \( w(k) \) has mean \( \nu \) and variance \( \sigma^2 \).
Norbert Wiener

Then the continuous time version has the form

$$d \ln S(t) = \nu \, dt + \sigma \, dz$$

Here $dz$ is a standard Wiener process. It behaves like a standard normal r.v.
Generalized Wiener Process

\[ d \ln S(t) = \nu \, dt + \sigma \, dz \]

The change in log-price of a stock is the result of a constant (the mean of w(k)) that gets perturbed by a N(0,1) r.v. that is proportional to the standard deviation of w(k)

A GWP can be solved explicitly for \( \ln S(t) \)

\[ \ln S(t) = \ln S(0) + \nu t + \sigma z(t) \]

Also called Geometric Brownian Motion
Stochastic Calculus

In Calculus we can express $d\ln S(t)$ as

$$d \ln S(t) = \frac{dS(t)}{S(t)}$$

When $S(t)$ is a random variable, however, the rules of regular calculus do not apply and we must turn to the mathematics of stochastic calculus.

There is a small adjustment to be made and...
Stock Price

Because Wiener functions are not functions of ordinary variables and $S(t)$ is lognormal

$$\frac{dS(t)}{S(t)} = \left( \nu + \frac{1}{2} \sigma^2 \right) dt + \sigma dz$$

We make a small substitution and

$$\frac{dS(t)}{S(t)} = \mu dt + \sigma dz$$
Black-Scholes Model

Developed under the assumption that price fluctuations of the underlying stock can be described by a stochastic process

$S$ is the price of the underlying stock and let it be governed by a geometric Brownian motion process over a time interval $[0, T]$

$$dS = \mu S dt + \sigma S dz$$
Black-Scholes Model

Suppose there is also a risk-free asset with an interest rate $r$ over the period $[0, T]$ and the value $B$ of the asset satisfies the ODE

$$dB = rBdt$$

Now consider a security which depends on the value of $S$ and $t$. Let $f(S, t)$ denote the price of this security at time $t$.
Black-Scholes PDF

Suppose that the price of a security is governed by the above and the risk-free rate $r$. A derivative of this security has a price $f(S, t)$, which satisfies the partial differential equation

$$\frac{\partial f}{\partial t} + \frac{\partial f}{\partial S} r S + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} \sigma^2 S^2 = rf$$
Black-Scholes Equation

There are lots of solutions to the Black-Scholes PDE – any one is a Black-Scholes Equation

\[ f(S, t) = S \quad \text{and} \quad f(S, t) = e^{rt} \]

By specifying the appropriate boundary conditions we find the right equation for the particular derivative (call, put, exotic)
Boundary Conditions for a Call

To find the price $C(S, t)$ of a European call option on a stock that pays no dividends must satisfy the Black-Scholes equation and satisfy the boundary conditions

$$C(0, t) = 0$$

$$C(S, T) = \max(S - K, 0)$$

Most options priced via simulation, but European options can be priced analytically.
Black-Scholes Call Option Formula

For a European call option with strike price $K$ and expiration time $T$. If the underlying stock pays no dividends during the time $[0, T]$ and if interest is constant and continuously compounded at a rate $r$, the Black Scholes solution is

\[
C(S, t) = SN(d_1) - Ke^{-r(T-t)}N(d_2)
\]
Black-Scholes Call Option Formula

where

\[ d_1 = \frac{\ln(S / K) + (r + \frac{\sigma^2}{2})(T - t)}{\sigma \sqrt{T - t}} \]

and \( d_2 = d_1 - \sigma \sqrt{T - t} \)

\( N(x) \) denotes the standard normal cumulative density function.
Example 13.2

Calculate the value of a 5 month call option on a stock with a current price of $62 and volatility of 20% per year. The strike price is $60 and the interest rate is 10%
Black-Scholes-Merton (Dalai Lama)