Diversification

• A portfolio with very few assets may be risky, as measured by variance.
• Variance of a portfolio can be reduced by adding more assets.
• The way portfolio variance is measured implies that assets with low covariance to each other can help lower overall variance.
• Extreme case: all assets uncorrelated.
Portfolio of Uncorrelated Assets

Suppose we have $n$ assets, equally weighted, uncorrelated, with mean $m$ and variance $\sigma^2$

Return of the portfolio is

$$r = \frac{r_1}{n} + \frac{r_2}{n} + ... + \frac{r_n}{n} = \frac{1}{n} \sum_{i=1}^{n} r_i$$

The expected value of this is $m$ and is independent of $n$
Variance of a Portfolio

Variance of the portfolio is

\[ Var(r) = \frac{1}{n^2} \sum_{i=1}^{n} \sigma_i^2 = \frac{\sigma_2}{n} \]

When assets are uncorrelated, portfolio variance can be made arbitrarily small.

When assets are correlated there may be a limit to how much variance can be lowered.
Variance of Correlated Assets

Assume each asset has rate of return with mean $m$ and variance $\sigma^2$, and each return pair has covariance $0.3\sigma^2$ for $i \neq j$.

The variance of the such a portfolio is

$$Var(r) = E\left[\sum_{i=1}^{n} \frac{1}{n} (r_i - \bar{r})\right]^2 = \frac{0.7\sigma^2}{n} + 0.3\sigma^2$$
Portfolio Diagram Lemma

The curve in an $r - \sigma$ diagram defined by non-negative mixtures of two assets 1 and 2 lies within the triangular region defined by the two original assets and the point on the vertical axis of height

\[
A = \frac{- (r_1 \sigma_2 + r_2 \sigma_1)}{(\sigma_1 + \sigma_2)}
\]
The Feasible Set

Feasible Set: The set of points corresponding to portfolios on a mean-variance diagram.

Minimum Variance Set: The left boundary of a feasible set.

Risk Averse Investor: Seeks to minimize risk (or s.d.); prefers the portfolio corresponding to the leftmost point on the line.

Risk Preferring Investor: Chooses a point other than one of minimum s.d.
Nonsatiation and Efficient Frontier

Nonsatiation: Investors always want the highest possible return for a given risk.

Efficient Frontier: The set of points in the minimum variance set that satisfy the nonsatiation constraint.

Efficient Portfolios: Provide the best mean-variance combinations.
Markowitz Model

Formulate a mathematical problem that leads to minimum-variance portfolios

Assume there are $n$ assets, each with expected rate of return $r_1, r_2, \ldots, r_n$ and covariances $\sigma_{ij}$

A portfolio has weights $w_i, i = 1, 2, \ldots, n$ sum to 1

To find the minimum variance portfolio, fix the expected return of the portfolio at an arbitrary value $r$ and solve for the feasible portfolio
Solving for the Feasible Portfolio

Has minimum variance: Minimize

\[
\frac{1}{2} \sum_{i,j=1}^{n} w_i w_j \sigma_{ij}
\]

Subject to

\[
\sum_{i=1}^{n} w_i r_i = r
\]

\[
\sum_{i=1}^{n} w_i = 1
\]
Solution Obtained by Using Lagrange Multipliers and Forming Langrangian

Step 1

Set each constraint equal to zero and multiply each by its respective Lagrange Multiplier

\[ \sum_{i=1}^{n} w_i r_i - r = 0 \quad \text{and} \quad \lambda \left( \sum_{i=1}^{n} w_i r_i - r \right) \]

\[ \sum_{i=1}^{n} w_i - 1 = 0 \quad \text{and} \quad \mu \left( \sum_{i=1}^{n} w_i - 1 \right) \]
Solution of the Markowitz Problem

Step 2

Form the Langrangian $L$

\[
L = \frac{1}{2} \sum_{i,j=1}^{n} w_i w_j \sigma_{ij} - \lambda \left( \sum_{i=1}^{n} w_i \bar{r}_i - \bar{r} \right) - \mu \left( \sum_{i=1}^{n} w_i - 1 \right)
\]
Solution of the Markowitz Problem

Step 3
Differentiate Langrangian with respect to each variable that we want to solve for, i.e.

$$w_1, w_2, \ldots, w_n$$

The result is $n$ partial derivatives plus the two original constraint equations
Solution of the Markowitz Problem

Set each partial derivative to 0

This results in $n$ equations plus 2 equations that formed the original constraints

We have $n+2$ equations for $n+2$ variables (we must solve for $\lambda$ and $\mu$ as well)

All of these equations are linear so we can use linear algebra to solve these equations
Equations for the Efficient Set

$$\sum_{j=1}^{n} \sigma_{ij} w_j - \lambda r_i - \mu = 0 \text{ for } i = 1, 2, \ldots, n$$

$$\sum_{i=1}^{n} w_i r_i = r$$

$$\sum_{i=1}^{n} w_i = 1$$
Equations for the Efficient Set

The $n$ portfolios weights $w_i$ for $i = 1, 2, ..., n$ and the two Lagrange multipliers $\mu$ and $\lambda$ for an efficient portfolio having mean return $\bar{r}$ satisfy

$$\sum_{j=1}^{n} \sigma_{ij} w_j - \lambda \bar{r}_i - \mu = 0 \quad \text{for } i = 1, 2, ..., n$$

$$\sum_{i=1}^{n} w_i \bar{r}_i = \bar{r}$$

$$\sum_{i=1}^{n} w_i = 1$$
Nonnegativity Constraints

If we restrict each portfolio to be positive, this leads to the additional constraint: \( w_i \geq 0 \)

This problem cannot be reduced to the solution of a set of linear equations.

It becomes a quadratic program because the objective function is quadratic and constraints are linear equalities and inequalities.

Big difference when shorting is allowed.
Example 6.10

Suppose we have three uncorrelated assets with means 1, 2, 3, respectively and variance 1. Find the weights of the minimum variance portfolio that has expected return. What happens if we apply a nonnegativity constraint to the portfolio weights?
Excel Matrix Multiplication

=mininverse( )
=mmult( )