INSTRUCTIONS: Do any 5 of the following 6 questions. All questions carry equal marks.

1. (20 marks) Prove by mathematical induction the identity
\[ \sum_{k=0}^{n} C(n, k) = 2^n. \]
Here, \( C(n, k) \) denote the binomial coefficients \( \frac{n!}{k!(n-k)!} \). Be sure to prove any identities used during your main proof.

SOLUTION

This was the Additional Question from the Advanced Calculus chapter 1 homework. See the online solutions to that homework set for the answer.

NOTE 1: in order to make the online solution to the above problem clearer, I have made a few minor changes to it. Go to the class website and click on Homework solutions then on Advanced Calculus, chapter 1 and Linear Algebra, chapter 1 solutions to see the updated answer to the above problem.

NOTE 2: For this problem, the result is true for \( n = 0, 1, 2, \ldots \), so in fact for the initial step, you should show it’s true for \( n = 0 \). Many of you showed it was true for \( n = 1 \) as the first step. I didn’t penalize anyone for this since the problem didn’t explicitly state the range of \( n \) values for which you should prove the theorem to be true, but note that in fact the result the result is also true for \( n = 0 \).

2. (a) (10 marks) Find a \( 2 \times 2 \) orthogonal matrix \( A \) whose first row is a (positive) multiple of \((3, 4)\).
(b) (10 marks) Find an equation of the plane \( H \) in \( \mathbb{R}^3 \) that contains the point \( P(1, -3, -4) \) and is parallel to the plane \( H' \) determined by the equation
\[ 3(x + 2) - 6(y - 2) + 5(z - 4) = 0. \]
(a) This solved problem 2.29 on page 51 of your textbook. Alternatively, you could do it the way I did 2.71(b) in a class example: Let \( \vec{u} = (3, 4) \) and \( \vec{v} = (x, y) \). We will try to choose \( \vec{v} \) such that some scalar multiple of \( \vec{v} \) will form the second row of our \( 2 \times 2 \) orthogonal matrix \( A \). We know that the rows of an orthogonal matrix must form an orthonormal set of vectors. In particular, we must have that \( \vec{u} \cdot \vec{v} = 0 \), or, more equivalently, \( 3x + 4y = 0 \). So \( y = -\frac{3}{4}x \). So, for example, letting \( x = 4 \), we get \( y = -3 \) so that the vector \( (x, y) = (4, -3) \) is orthogonal to \( \vec{u} = (3, 4) \). The next step is to normalize these vectors (multiply them by the reciprocal of their norms) so that we get two unit vectors. Since \( |\vec{u}| = \sqrt{9 + 16} = 5 \) and \( |\vec{v}| = \sqrt{16 + 9} = 5 \), we get

\[
\frac{1}{|\vec{u}|} \vec{u} = \left( \frac{3}{5}, \frac{4}{5} \right)
\]

and

\[
\frac{1}{|\vec{v}|} \vec{v} = \left( \frac{4}{5}, \frac{-3}{5} \right).
\]

And the desired orthonormal matrix is:

\[
A = \begin{bmatrix}
\frac{3}{5} & \frac{4}{5} \\
\frac{-4}{5} & \frac{-3}{5}
\end{bmatrix}
\]

(b) This is solved problem 1.18 (page 18) of the Linear Algebra textbook, \textit{and it was also an assigned problem}. I’ve simply written the equation of the plane \( H' \) slightly differently to the way it is written in problem 1.18 (in point-normal form instead of standard form).

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3. (a) \(10 \) marks) Let \( A \) and \( B \) be upper triangular \( n \times n \) matrices. Prove that \( AB \) is also upper triangular, with main diagonal entries \( a_{11}b_{11}, a_{22}b_{22}, \ldots, a_{nn}b_{nn} \).

(b) \(10 \) marks) Find a parametric representation of the line \( L \) in \( \mathbb{R}^4 \) which passes through the two points \( P_1(1, -2, 3, -4) \) and \( P_2(3, -5, 7, 9) \).
(a) This is solved problem 2.25 (page 49) of the Linear Algebra textbook, and it was also an assigned problem.

**NOTE** There is one point in the textbook’s proof which I want to make clearer. Given

\[(AB)_{ij} = c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj},\]

and supposing \(i > j\), clearly \(a_{ik} = 0\) (hence the above sum is 0) whenever \(i > k\) (since A is upper triangular). However, what is not made as clear in the text is that whenever the opposite holds, *i.e.* \(k \geq i\), the reason we know that \(k > j\) (and hence \(b_{kj} = 0\) since \(B\) is upper triangular) is as follows: Recall \(i > j\). Hence, if \(k \geq i\), then \(k \geq i > j \Rightarrow k > j\).

(b) This question is similar to an example done in class. Recall that we want a vector parallel to the line, and one such vector is \(\vec{P}_1\vec{P}_2 = (3 - 1, -5 + 2, 7 - 3, 9 + 4) = (2, -3, 4, 13)\). We now have a vector parallel to the line and a choice of two points, \(P_1\) and \(P_2\), on the line. So a parametric equation of the line (using the point \(P_1\)) is \(\{(x_1, x_2, x_3, x_4) : x_1 = 1 + 2t, x_2 = -2 - 3t, x_3 = 3 + 4t, x_4 = -4 + 13t\}\), where \(-\infty < t < +\infty\).

4. (20 marks) A square matrix \(A\) is **nilpotent** if \(A^k = 0\) for some positive integer \(k\). Show that if \(A_{n\times n}\) and \(B_{n\times n}\) are both nilpotent and \(AB = BA\), then \(A + B\) is also nilpotent. Point out where in your proof you use the fact that \(A\) and \(B\) commute (i.e., that \(AB = BA\)).

**SOLUTION**

Note that if \(AB = BA\), then \(A^m B^n = B^n A^m\) for any positive integers \(m\) and \(n\). To see this, just observe that \(A^m B^n = AA \ldots ABB \ldots B = BA^m B^{n-1}\) by successive use of the commutativity of \(A\) and \(B\) on terms of the form \(AB\) to move the \(B\) term one place to the left. Repeating the argument above for the other \(n - 1\) \(B\)’s we get the stated result. (A “proper” way to prove that result is by induction on one of the indices, \(m\) or \(n\). But given that the result is fairly obvious and this is
an exam. and you have limited time, you will not be penalized for not giving a more rigourous proof).

Note also if $C^k = 0$, then $C^{k+l} = 0$ also for any positive integer $l$. This is because $C^{k+l} = C^k C^l = 0 C^l = 0$ since the product of any $n \times n$ matrix with the $n \times n$ zero matrix is the $n \times n$ zero matrix.

Next, note that if $AB = BA$ and therefore $A^m B^n = B^n A^m$, then the binomial theorem (proven as homework problem 1.95 in the Linear Algebra textbook) applies to matrices also. So

$$(A + B)^k = A^k + k C_1 A^{k-1} B + k C_2 A^{k-2} B^2 + \ldots + k C_{k-1} A B^{k-1} + B^k = \sum_{l=0}^{k} k C_l A^l B^{k-l}$$

for any positive integer $k$ (where the $p C_q$ are binomial coefficients). NOTE, it is here in the proof we use the commutativity of $A$ and $B$, in order to combine (the coefficients of) terms of the form $A^l B^{k-l}$ and terms of the form $B^{k-l} A^l$.

Since $A$ and $B$ are by assumption nilpotent, let $k_1$ be the smallest positive integer such that $A^{k_1} = 0$ and let $k_2$ be the smallest positive integer such that $B^{k_2} = 0$. Then, I claim that $(A + B)^{k_1 + k_2} = 0$, meaning that $A + B$ is also nilpotent.

**Proof of Claim:** $(A + B)^{k_1 + k_2} = \sum_{l=0}^{k_1 + k_2} k_1 k_2 C_l A^l B^{k_1 + k_2 - l}$.

- Now if $l \geq k_1$, then all $A^l$ terms are the 0 matrix, since $A$ is nilpotent. Hence all terms involving the matrix $A^l$ will simplify to the 0 matrix.

- Next, if $l < k_1$, then $k_1 + k_2 - l > k_2$, so that, since $B$ is nilpotent, all $B^{k_1 + k_2 - l}$ terms are the 0 matrix. Hence all terms involving the matrix $B^{k_1 + k_2 - l}$ will simplify to the 0 matrix.

In summary, we have shown that $(A+B)^{k_1 + k_2}$ is just a sum of 0 matrices hence is simply the 0 matrix.

**Note 1:** in the above proof, the value of the binomial coefficients did not play an important role. So you don’t really need the binomial theorem for this proof. All you need to know is that $(A + B)^k = \sum_{l=0}^{k} d_k A^l B^{k-l}$ for some constants $d_k$, and a more precise expression for those constants is not really important to the proof.
NOTE 2: The requirement that $A$ and $B$ commute is essential. It is not generally true that $A + B$ is nilpotent if both $A$ and $B$ are nilpotent and $AB \neq BA$. For example, if $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, then both $A$ and $B$ are nilpotent. But $A + B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is not nilpotent since $(A + B)^{even\ number} = I$ and $(A + B)^{odd\ number} = A + B$.

5. Let $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 3 \\ -3 & -10 & 2 \end{bmatrix}$.

(a) (10 marks) Find the $LU$ factorization of $A$.

(b) (10 marks) Let $x_k$ denote the solution of $A x = b_k$, $(k = 1, 2, \ldots)$. Find $x_1$ and $x_2$ when $b_1 = (1, 1, 1)^T$ and $b_2 = b_1 + x_1$.

SOLUTION

This is (part of) solved problems 3.41 and 3.42 (pages 109 - 110) of the Linear Algebra textbook, and they were also assigned problems.

NOTE: the textbook solution has one minor error. The solution vector $x_2 = (977, -353, -295)$, not $x_2 = (943, -353, -295)$.

6. Consider the following system of linear equations in the unknowns $x$, $y$, and $z$:

$$
\begin{align*}
 x + y + 2z &= 1 \\
 2x + 4y + az &= 2 \\
 x + ay + 4z &= b
\end{align*}
$$

(a) (10 marks) For which values of $a$ does the system have a unique solution?

(b) (10 marks) For which pairs of values $(a, b)$ does the system have more than one solution?

HINT: Depending on how you do this problem, it might seem as if there are three values of $a$ for which the system does NOT have a unique solution. But in fact you should be able to rule out one of these values and get only two values of $a$ for which the system does NOT have a unique solution.
SOLUTION

First, reduce the augmented matrix form of the system of equations to row echelon form:

\[
\begin{bmatrix}
1 & 1 & 2 & | & 1 \\
2 & 4 & a & | & 2 \\
1 & a & 4 & | & b
\end{bmatrix}
\]

\[R_2 \mapsto R_2 - 2R_1 \sim \begin{bmatrix}
1 & 1 & 2 & | & 1 \\
0 & 2 & a - 4 & | & 0 \\
1 & a & 4 & | & b
\end{bmatrix}\]

\[R_3 \mapsto R_3 - R_1 \sim \begin{bmatrix}
1 & 1 & 2 & | & 1 \\
0 & 2 & a - 4 & | & 0 \\
0 & a - 1 & 2 & | & b - 1
\end{bmatrix}\]

\[R_2 \mapsto (a - 1)R_2 \sim \begin{bmatrix}
1 & 1 & 2 & | & 1 \\
0 & 2(a - 1) & (a - 1)(a - 4) & | & 0 \\
0 & -2(a - 1) & -4 & | & -2(b - 1)
\end{bmatrix}\]

\[R_3 \mapsto -2R_3 \sim \begin{bmatrix}
1 & 1 & 2 & | & 1 \\
0 & 2(a - 1) & (a - 1)(a - 4) & | & 0 \\
0 & 0 & (a - 1)(a - 4) - 4 & | & -2(b - 1)
\end{bmatrix}\]

\[R_3 \mapsto R_3 + R_2 \sim \begin{bmatrix}
1 & 1 & 2 & | & 1 \\
0 & 2(a - 1) & (a - 1)(a - 4) & | & 0 \\
0 & 0 & (a - 1)(a - 4) - 4 & | & -2(b - 1)
\end{bmatrix}\]

But since \((a - 1)(a - 4) - 4 = a^2 - 5a + 4 - 4 = a^2 - 5a = a(a - 5)\), we may write this last matrix as

\[
\begin{bmatrix}
1 & 1 & 2 & | & 1 \\
0 & 2(a - 1) & (a - 1)(a - 4) & | & 0 \\
0 & 0 & a(a - 5) & | & -2(b - 1)
\end{bmatrix}
\]

(a) We recall that the system has a unique solution if the coefficient matrix portion (i.e., everything except the last column) of this row echelon form of the matrix has as many pivots as equations : 3. So, in particular, we require that the entry in row 3 column 3 be nonzero if the system is to have a unique solution. So

\[a \neq 0, 5.\]

NOTE: it might be tempting to say that the pivot entry in row 2 column 2, \(2(a - 1)\), could be zero if \(a = 1\) (and so there would be no unique solution if \(a = 1\)). However, note that if \(a = 1\), we cannot perform the row operation \(R_2 \mapsto (a - 1)R_2\) since it would involve multiplying a row by 0 (remember that the row scaling elementary row operation only allows multiplying a row by a nonzero scalar). So, instead, we can substitute \(a = 1\) into the augmented matrix in the
following stage of Gaussian elimination

\[
\begin{bmatrix}
1 & 1 & 2 & 1 \\
0 & 2 & a - 4 & 0 \\
0 & a - 1 & 2 & b - 1
\end{bmatrix}
\]

This will automatically give us a row echelon form of the matrix which has 3 pivots, hence the system has a unique solution when \(a = 1\).

(b) You want the entire last row of the row echelon form of the system of equations to be zero. So the answer is

\[(a, b) = (0, 1) \text{ and } (5, 1)\].