Section 6.1 Generating Function models

Def: Suppose $a_r$ is the number of ways to select $r$ objects in some procedure. Then $g(x)$ is a generating function for $a_r$ if

$$g(x) = a_0 + a_1x + a_2x^2 + \cdots a_rx^r + \cdots$$

Example:

$$(1 + x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots \binom{n}{r}x^r + \cdots = \binom{n}{n}x^n$$

Example:

$$(1 + x + x^2)^4$$

Example 1: Find the generating function for $a_r$ the number of ways to select $r$ balls from 3 green, 3 white, 3 blue, and 3 gold.

Example 2: Find the generating function for $a_r$ the number of ways to select $r$ doughnuts from 5 chocolate, 5 strawberry, 3 lemon, 3 cherry? Reapeat with at least 1 per type.

Example 3: Find the generating function for $a_r$ the number of ways to select $r$ objects from 3 types with at most 4 per type. What if there is no limit on the number from each type?

Example 4: Find the generating function for $a_r$ the number of ways to distribute $r$ identical objects into 5 distinct boxes, with an even number of objects not exceeding 10 in the first two boxes, and between 3 and 5 in the other boxes.
Exc 2(b): Find the generating function for \( a_r \) the number of integer solutions to \( e_1 + e_2 + e_3 = r \) with \( 0 < e_i < 5 \).

Exc 3(c): Find the number of selection of \( r \) objects from unlimited piles of pennies, nickels, dimes and quarters.

Exc 9: Find the number of \( r \) combinations of an \( n \) set with repetition allowed.

Exc 8(a): Build the generating function for the number of election outcomes for class president with 4 candidates and 27 voters. Which coefficient solves the problem?

(b). Same, but each candidate votes for him/herself.

(c). Same as in (a), but no candidate receives a majority of votes.

Exc 22: Find the generating function for \( a_r \) the number of integer solutions to \( e_1 + e_2 + e_3 + e_4 = r \) with \( 0 \leq e_1 \leq e_2 \leq e_3 \leq e_4 \).
Section 6.2 Calculating coefficients of Generating Functions

\[
\frac{1 - x^{n+1}}{1 - x} = 1 + x + x^2 + \cdots + x^n \quad (1)
\]

\[
\frac{1}{1 - x} = 1 + x + x^2 + x^3 + \cdots \quad (2)
\]

\[
(1 + x)^n = 1 + \binom{n}{1} x + \binom{n}{2} x^2 + \cdots + \binom{n}{r} x^r + \cdots + \binom{n}{n} x^n \quad (3)
\]

\[
(1 - x^m)^n = 1 - \binom{n}{1} x^m + \binom{n}{2} x^{2m} + \cdots + (-1)^r \binom{n}{r} x^{rm} + \cdots + (-1)^n \binom{n}{n} x^{nm} \quad (4)
\]

\[
\frac{1}{(1 - x)^n} = 1 + \binom{1 + n - 1}{1} x + \binom{2 + n - 1}{2} x^2 + \cdots + \binom{r + n - 1}{r} x^r + \cdots \quad (5)
\]

\[
(a_0 + a_1 x + \cdots) \times (b_0 + b_1 x + \cdots) = a_0 b_0 + (a_1 b_0 + a_0 b_1) x + (a_2 b_0 + a_1 b_1 + a_0 b_2) x^2 + \cdots \quad (6)
\]

Equation (6) can be restated: \(c_k = a_k b_0 + a_{k-1} b_1 + \cdots + a_0 b_k\)

Example 1: Let \(a_r\) be the number of ways to distribute \(r\) identical objects into 5 distinct boxes, with at least 2 per box. What is \(a_{16}\), the coefficient of \(x^{16}\)?

Example 2: What is the number of ways to collect $15 from 20 people if the first 19 people each give $0 or $1, and the 20th person gives $0, $1 or $5.

Example 3: What is the number of ways to distribute 25 identical balls into 7 distinct boxes with box 1 having at most 10 balls?

Exc 17: What is the number of ways to select 10 balls out of Red, White, and Blue balls:
(a) At least 2 of each colour?
(b) At most 2 red balls?
(c) An even number of red balls?
(d) An even number of every colour?

Example 4: What is the number of ways to select 25 objects from 7 types, with between 2 and 6 of each type.

Example 5: Verify:

\[
\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}
\]