AMS 301.3 (Fall, 2009) Estie Arkin

Exam 1 – Solutions

Mean 76.68, median 80, high 96, low 38, all scores out of 96 points.

1. (14 points) Are the two graphs shown below isomorphic? If so, give the isomorphism; if not, give careful reasons for your answer. No, they are not isomorphic. Graph on the left is bipartite, graph on the right is not, as it has an odd circuit (1-2-6-8-5-1). (Or: graph on left is planar, right one is not. This requires proof.)

2. (14 points) Compute the chromatic number of the graph G shown below. Justify your answer! (Show a coloring with $\chi(G)$ colors (label each node with its color), and argue that fewer colors cannot suffice.)

There are several odd circuits, the graph is not bipaprtite (A,B,C,D,E,A) so at least 3 colours are needed. There are many 3 colourings, such as: A=1, B=3, C=2, D=1, E=2, F=2, G=1, H=3, I=2, J=3.

Common mistakes: Saying there is a K_5 or a 5 wheel, or that since the degree at each node is 3 then 3 colours are needed (which is true if colouring edges, not nodes!).

3. (20 points) True or False? If true, give a short proof. If false, give a counterexample:

(a). Every complete bipartite graph $K_{n,m}$ is connected ($n, m \geq 1$). True. Every node "on the right" has an edge to "every node on the left", so these are connected by a path of length 1. Also, every node on the right is connected by a path of length 2 to every other node on the right, going through a node, say node 1, on the left. Similarly, every node on the left is connected by a path of length 2 to every other node on the left.

Common mistake: Saying that complete bipartite graphs have an edge between *every* pair of nodes. It does not have edges between nodes on the same side!

(b). The *edges* of a planar graph can be coloured by at most 5 colours. False. The graph $K_{1,10}$ is planar but reguires 10 diferent colours for the edges.

Common mistakes: Confusing edge colouring and node colouring (the nodes of a planar graph can be coloured by 5 colours, but that is not the question here!). Giving a graph $K_{10,16}$ which is not planar as an example.

(c). For every bipartite graph G, its complement must also be bipartite. False. The complement of $K_{3,3}$ is comprised of two disjoint K_3s, and therefore is not bipartite.

Note: The complement of $K_{1,5}$ is not K_5! It must have 6 nodes, just like $K_{1,5}$ does. The complement is an isolated node plus K_5.

(d). If G is a graph in which all nodes have the same degree, then its complement must also have all nodes of the same degree.

True. Let k be the degree of all nodes in G, and let n be the number of nodes. As we have seen in class, the degree of node i in G plus the degree of node i in the complement is equal to $n-1$ the degree in a complete graph. Thus, each node of the complement graph must have degree $n-1-k$.

(e). Let G be a graph that has a Hamilton circuit, then G must also have an Euler cycle. False. K_4 has a Hamilton circuit but no Euler cycle.

4. (12 points) Model the following problem as a graph colouring problem: A school is trying to schedule review sessions for upcoming regents exams in 11 different subjects (math A, math B,

english, spanish, world history, etc.). Each subject should have one review session, and each review session should be 3 hours long and will be held either 8-11am or 1-4pm on one of the 4 day school vacation next week. Each student provides the guidance counselors with a list of subjects they plan to attend review sessions for. Some students may need to attend review sessions in several subjects, so such review sessions cannot be scheduled at the same time.

(a). What exactly does the set of vertices V correspond to? Review sessions for different subjects (regents exams).

(b). What exactly does the set of edges E correspond to? A student who wants to attend both review sessions.

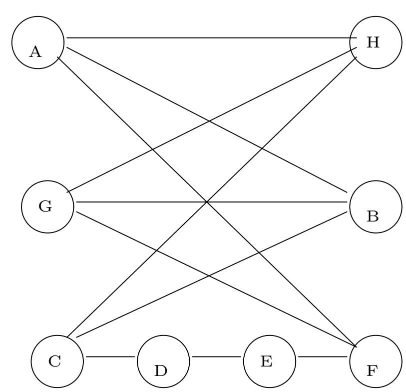
(c). What is being coloured, the vertices or the edges? Vertices.

(d). You are told that the graph you defined can be coloured with 6 colours. Can all review sessions be scheduled without conflicts? Explain! There are 8 different times slots, (4 morning and 4 afternoon) and 6 are needed to avoid all conflicts, so it is possible $6 \leq 8$.

Common mistakes: Claiming the graph is planar so is 6 colourable. The graph may or may not be planar, but we are given that it is 6 colourable. Another mistake: saying maybe there is a student that needs to attend more than 8 review sessions. If that were the case, then the graph would not have been 6 colouravle, as we are given!

5. (14 points) Show that the following graph is non planar by showing the $K_{3,3}$ or K_5 configuration it contains.

There are many $K_{3,3}$ configurations such as:



6. (12 points) A planar connected graph G has 10 nodes and 20 edges.

(a). How many regions does the graph G have? $r = e - v + 2$, so $r = 20 - 10 + 2 = 12$.

(b). How many edges does the complement of this graph, $\bar{G}$ have? The complete graph on 10 nodes has $10 \cdot 9/2 = 45$ edges. As we have seen in class, the number of edges in G plus the number of edges in its complement is equal to the number of edges in the complete graph. Thus the complement graph has $45 - 20 = 25$ edges.

(c). Is it possible that $\bar{G}$ is planar? Explain! We check: Is $\bar{e} \leq 3v - 6$? ($\bar{e}$ denotes the number of edges in the complement graph). No. Thus by the corollary to Euler's formula, the complement graph is not planar.

7. (14 points) Prove that no Hamilton circuit exists in the following graph:

Use symmetry at node K, 2 of the 3 edges touching K must be used, and the choice is symmetric, so assume (K,I) and (K,C) are used, and (K,F) is removed. Rule 1 at F, use edges (E,F) and (F,G).

Case 1: Neither (A,E) nor (B,G) are used: Rule 1 at G and E so (G,H) and (E,D) are used. By rule 2, cannot use (D,H). Rule 1 at H and D, subcircuit is formed (K,C,D,E,F,G,H,I,K).

Case 2: At least one of (A,E) or (B,G) are used (possibly both). By symmetry it doesn't matter which, so without loss of generality, say (A,E). Rule 3 at E, no (E,D). Rule 1 at D so use (D,C) and (D,H). Rule 2 no (I,H). Rule 1 at I and at H closes a subcircuit (A,I,K,C,D,H,G,F,E,A) violating rule 2.

Common mistakes:

Using Dirac's Theorem in the wrong direction: If a graph on n nodes has some (or all) nodes of degree less than $n/2$ this does not imply there is no Hamilton circuit. For example, a circuit on 1,000,000 has all nodes of degree 2 (which is much less than 500,000) but does have a Hamilton circuit.

Saying that the graph is not bipartite and therefore has no Hamilton circuit: While this graph is not bipartite, that does not tell us whether it does or does not have a Hamilton circuit. For example K_3 is not bipartite, but does have a Hamilton circuit.

Saying that by rule 2 one or more of the "external" edges cannot be used, and all such edges are symmetric, so remove (A,B): Not all the external edges are symmetric, in particular (A,B) is different fro (A,I). Also, once some choices are made, the remaining graph may no longer be symmetric, so cannot then use the original symmetry at node K. It must be used at the start (if at all).

Using Theorem 0 incorrectly: Removing 3 nodes C,F,I results in only 2 connected components not 4. Node K is one connected component, and nodes A,B,D,E,G,H together are a second connected component.

Remember: Every time you say "choose edge .. to use" or "to remove" there should be an explanation why you are making such a choice, and what would happen if you mae a different choice.