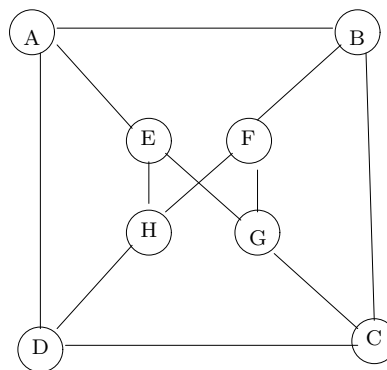
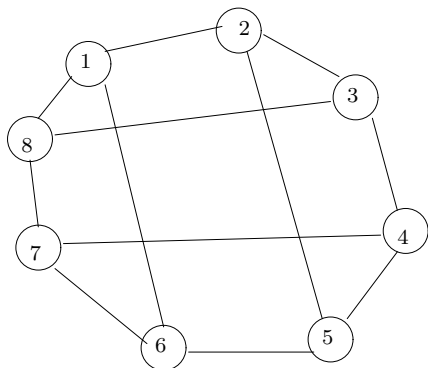


Exam 1 – Solutions

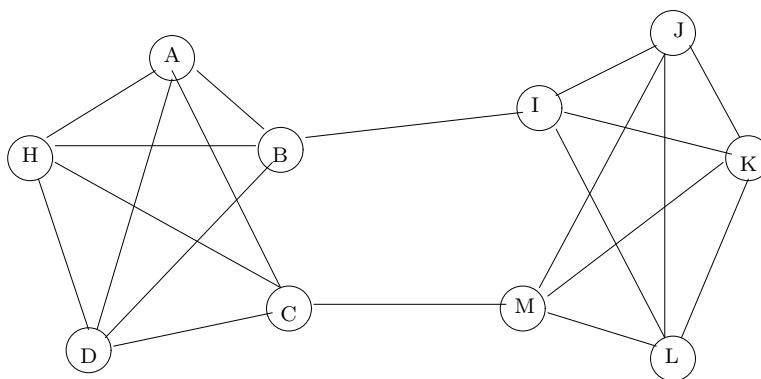
Mean 77.97, median 82.5, top quartile 91, high 100 (2 of them), low 34.

1. (13 points) Are the two graphs shown below isomorphic? If so, give the isomorphism; if not, give careful reasons for your answer.



No. The graph on the left is bipartite (nodes 1,3,5,7 on one side and nodes 2,4,6,8 on the other) but the graph on the right is not bipartite, it has an odd circuit A,E,G,C,B,A (5 nodes). Also, the graph on the left is planar, the graph on the right is not.

2. (12 points) Compute the chromatic number (vertex colouring number) of the graph G shown below. Justify your answer! (Show a colouring with $\chi(G)$ colours (label each node with its colour), and argue that fewer colours cannot suffice.)



There are several K_4 s such as nodes A,C,D,H so at least 4 colours are needed. 4 colours are also enough (the graph is planar) for example colour 1 are nodes B,C,K; colour 2 nodes A,I,M; colour 3 nodes H,L; colour 4 nodes D,J.

3. (25 points) True or False? If true, give a short proof. If false, give a counterexample:

(a). Every subgraph of a bipartite graph is also bipartite. True. The nodes of the graph are from two sets “left” and “right”, and the subgraph selects some of the left and some of the right nodes, and some of the edges. There will be no edges between two left nodes or between two right nodes.

Common mistakes: Saying that a graph must be connected to be bipartite., or saying that if the subgraph selects only nodes on one side, then it is not bipartite. If a graph has no edges, then it is trivially bipartite. Another common mistake: Saying that if a graph has no circuits then it cannot be bipartite, since all circuits must be of even length. If a graph has NO circuits, then again, it is by definition, bipartite!

(b). Every subgraph of a complete graph is also a complete graph. False. Consider K_3 , and a subgraph that includes all three nodes but only 2 of the edges. It is not complete.

Common mistake: Removing only nodes. In a subgraph, as defined in class and textbook, we are allowed to remove nodes and/or edges.

(c). Let H be a subgraph of G , then $\chi(H) \leq \chi(G)$. True, you can colour the nodes of H by the same colours as in G . There may be a better colouring.

(d). If G contains a K_5 configuration then $\chi(G) \geq 5$. False. Subdivide each edge of K_5 by placing exactly one new node along the edge. The resulting graph is bipartite, so requires only 2 colours! Another counterexample is the graph in problem 3 which can be coloured by 4 colours, but has a K_5 configuration.

(e). Let G be a graph that has an Euler cycle, then G must also have a Hamilton circuit. False. Consider the “bow tie” graph, figure 2.4 page 57. Common mistake: Giving an example that has a Hamilton circuit but not an Euler cycle.

4. (12 points) Model the following problem as a graph colouring problem: Stony Brook students are trying to schedule meetings of 12 student committees. Each meeting is to be held during Wednesday’s campus life time 12:50-2:10. Several students are very active and are part of more than one committee. The schedule should be arranged so that such students can attend all meetings of committees to which they belong.

(a). What exactly does the set of vertices V correspond to? Committees.

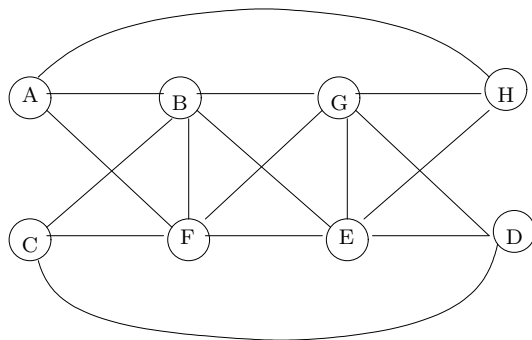
(b). What exactly does the set of edges E correspond to? Each edge corresponds to those 2 committees having at least one student member in common.

(c). What is being coloured, the vertices or the edges? Vertices. The colours are the week of the meeting.

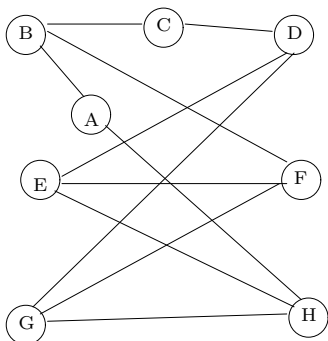
(d). You are told that the graph you defined can be coloured with 3 colours. Can all committee meetings be scheduled within the month of February, without conflicts? Explain! Yes. Since a month has 4 (or more) Wednesdays and we only need 3 Wednesdays for a schedule with no conflicts.

Common mistakes: Trying to use the Art Gallery Theorem, or saying that since $12/3=4$, we can schedule 3 meetings each week. We do not know this, as it depends on which students belong to which committees!

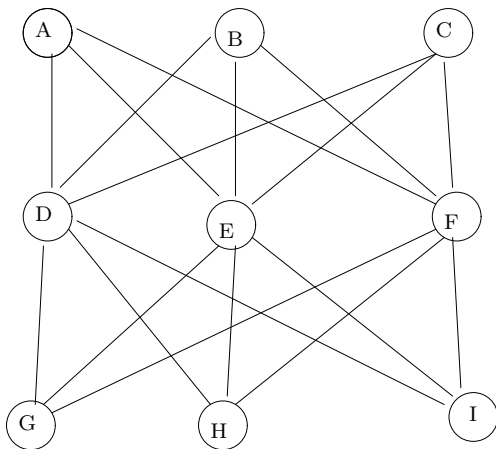
5. (13 points) Show that the following graph is non planar by showing the $K_{3,3}$ or K_5 configuration it contains.



We use the circle chord method to help us find the configuration. A Hamilton circuit is: A,B,C,D,E,F,G,H,A. The chords are (B,F) (D,G) (E,H). We get the following $K_{3,3}$ configuration:



6. (12 points) A planar connected graph G has 10 nodes each of degree 4.
- How many edges does the graph G have? $2e = 10 \cdot 4$ so $e = 20$.
 - How many regions does the graph G have? $r = e - v + 2 = 20 - 10 + 2 = 12$.
 - Is it possible that G is bipartite? Explain! No. For a planar bipartite graph, we have $e \leq 2v - 4$, and here we have $20 = e > 2 \cdot 10 - 4 = 16$.
7. (13 points) Prove that no Hamilton circuit exists in the following graph:



The graph is bipartite with nodes A,B,C,G,H,I on one side (6 nodes) and D,E,F on the other side (3 nodes) so by the theorem about bipartite graphs, no Hamilton circuit exists. Similarly we can say that removing 3 nodes D,E,F, creates 6 connected components, since the remaining 6 nodes are each a separate connected component, then by Theorem 0 there is no Hamilton circuit.

We can also use the three rules to show this. Note that the graph in the figure is $K_{3,6}$ and so there is a lot of symmetry. Node A must use 2 out of the 3 edges touching it, and all 3 edges are symmetric, so without loss of generality, we can assume that (A,D) and (A,E) are used, and then by rule 3 (A,F) can be removed. Now, by rule 2, we know that we cannot use both edges (B,D) and (B,E) as they would form a subcircuit. Again these edges are symmetric, so we can remove one of them without loss of generality, say we remove (B,D). Thus we must use (B,E) and (B,F) by rule 1 at B. Now use rule 3 at E, since E is already used by 2 edges in the Hamilton circuit, so we must remove (E,C), (E,G) (E,H) (E,I). Now node G,(as well as H,I) has degree 2, so we to apply rule 1 to node G, which means we must use edges (G,D) (G,F). Now rule 3 at D, remove (D,H) (D,I) and now nodes H and I have degree 1, so no Hamilton circuit exists.

Common mistakes: Removing edges without explaining why (or giving an incorrect reason), using a rule when it does not apply (such as rule 1 can only be used if a node has degree 2). Another mistake: Using Dirac's Theorem in the wrong direction. Just because the nodes do not have degree $\geq 9/2$ does NOT imply no Hamilton Circuit.