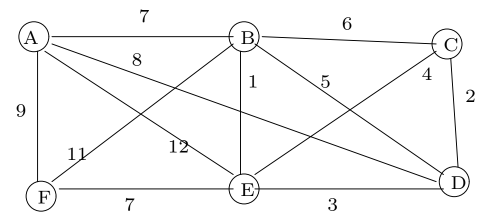
AMS 301.3 (Fall, 2009) Estie Arkin

Exam 2 – Solution sketch

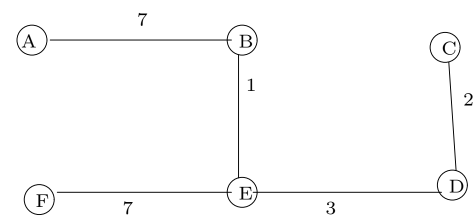
Mean 81.2, median 84.5, high 100 (4 of them!), low 25.

1. (5 points) Let T be a 6-ary tree with 100 internal nodes. How many leaves does the tree have? (A correct guess with no work shown will recieve very partial credit.) $n = mi + 1$, $m = 6$, $i = 100$, so $n = 601$, and $l = n - i = 501$.

2. (15 points) Consider the following graph.



(a). Highlight the edges of a minimum spanning tree of the graph. Using Kruskal's algorithm, egdes would be put into the MST in the following order: (B,E) (C,D) (D,E) (B,A) (E,F). The cost of this tree is 20.

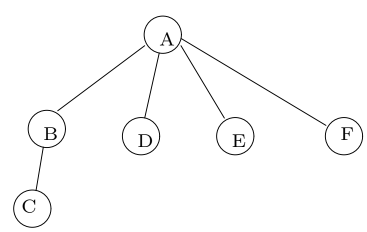


(b). The cost of every edge in the graph is multiplied by 10. Circle the correct answers:

Does this change the edges in the minimum spanning tree? NO, Kruskal's algorithm would put the edges in the tree in the same order.

Does this change the cost of the minimum spanning tree? YES, the new minimum cost is 10 times the old min cost.

(c). Ignoring the costs on the edges, show the BFS tree of the graph starting at node A.



Common mistake: drawing the MST again, or doing DFS.

3. (8 points) We wish to model the word problem as a graph problem: The Happy Gardner, a lawnmowing service, has been contracted to mow the lawns at 15 houses today. The gardener would like to plan his schedule for the day, namely, in what order the will the different lawns be

mowed. We may assume the gardener starts his day at his house, and would like to return home as soon as possible. We also assume travel times between houses are known.

(a). What do the nodes of the graph to be constructed represent? The houses (15 to be mowed and gardnere's house).

(b). What do the edges of the graph to be constructed represent? Travel between 2 houses. Common mistake: saying edges are times or distances to travl.

(c). State which graph problem it is: Traveling Salesman Problem (TSP).

4. (12 points) True or False? If true, give a short proof. If false, give a counterexample: We are given a connected graph G with costs on edges. Assume all costs are positive and that there are no ties.

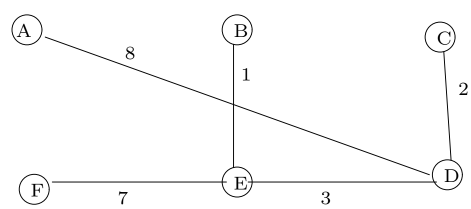
(a). If edge $e = (A, B)$ is the cheapest edge touching node A, then it must be part of a Minimum Spanning Tree. True, in fact it is the first edge that Prim (version from class, not text) would select. It may NOT be the first edge that Kruskal selects, since it may not be the cheapest edge in the graph. That could be some other edge (C,D).

(b). If edge $e = (A, B)$ is the cheapest edge touching node A, then it must be part of a shortest path tree rooted at node A. True, since the shortest path to B from node A is just this edge. Any other path will use at least one other edge, and that has cost more than (A,B). Also, Dijkstra's algorithm will make node B permanent first. Note many people simply restated the question: The cheapest edge must be used because it is the cheapest edge. This is not a proof.

(c). If edge $e = (A, B)$ is the cheapest edge touching node B, then it must be part of a shortest path tree rooted at node A. True. Any path from A to B must contain an edge touching B, and therefore must have cost that is $\geq$ the cost of edge e, and so e is the shortest path from A to B.

(d). If edge $e = (B, C)$ is part of a Minimum Spanning Tree then it must also be part of a shortest path tree rooted at node A. False. Consider the graph on nodes A,B,C with edges (A,B) and (A,C) of cost 2 and edge (B,C) of cost 1. The shortest path tree rooted at A contains edges (A,B) and (A,C) not (B,C). However (B,C) is part of the MST.

5. (10 points) (a). Highlight the edges of the shortest path tree rooted at node D for the graph below. The edges in the shortest path tree are (D,C) (D,E) (E,B) (D,A) (E,F).



Dijsktra's algorithm calculates:

u_A	u_B	u_C	u_D	u_E	u_F	node becoming perm
8	5	2	0	3	∞	(D and) node C
8	5	-	-	3	∞	node E
8	4	-	-	-	10	node B
8	-	-	-	-	10	node A
-	-	-	-	-	10	node F

The final pred labels are: $\text{pred}_A = D$, $\text{pred}_B = E$, $\text{pred}_C = D$, $\text{pred}_E = D$ $\text{pred}_F = E$.

(b). The cost of every edge in the graph is multiplied by 10.

Does this change the edges in the shortest path tree? NO

Does this change the cost of the shortest path from node D to node A? YES the cost would be 10 times larger.

6. (10 points) There are 100 (distinct) students at a school and 3 dormitories, A,B,C, with capacities 25, 35, and 40 respectively.

(a). How many ways are there to fill the dormitories? $\binom{100}{25}\binom{75}{35}\binom{40}{40} = P(100; 25, 35, 40)$. See the second part of example1 in section 5.4 discussed in class.

Common mistake: $P(100, 25)P(75, 35)P(40, 40)$ that would assume the order of people in the dorm is important.

(b). Suppose that of the 100 students 50 are men and 50 are women and that A is an all-men's dorm, B is an all-women's dorm and C is co-ed. How many ways are there to fill the dormitories? $\binom{50}{25}\binom{50}{35}\binom{40}{40}$. We select 25 of the 50 men to place in dorm A (order not important, repetition not allowed). Then we select 35 out of the 50 women to place in dorm B. The remaining 40 people are placed in dorm C.

(7).(5 points) Determine the number of ways to distribute 10 identical orange drinks, 1 lemon drink, and 1 lime drink to 4 thirsty (distinct!) children so that each child gets at least one drink, and the lemon and lime drinks go to different children.

$4 \cdot 3 \cdot \binom{8+4-1}{8}$, give the lemon drink to one child (4 choices) given the lime drink to a different child (3 choices) give 1 orange drink to each of the remaining 2 children, and then distribute the remain 8 orange drinks to the 4 children. Note: Since the orange drinks are identical, they are not distirbuted 4^8 ways, which is the formula for distribution of distinct items.

(8). (15 points) (a). How many distinct arrangements of the letters in "PREPOSTEROUS" are there? $P(12; 2, 2, 2, 2, 2, 1, 1)$.

(b). How many distinct arrangements of the letters in "PREPOSTEROUS" are there, in which the first "E" precedes the first "O"? $\binom{12}{4}\binom{3}{1}P(8; 2, 2, 2, 1, 1)$. We first choose 4 spots for the 2 E's and 2 O's. In the first of these 4 spots we place an E. Now arrange those remaining 3 spots with EOO. Finally arrange the remaining letters. An alternative approach: half of the arrangements from part (a) have the first E before the first O, and the remaining half have the first O before the first E, by symmetry (same number of E's and O's). So the answer is $0.5 \cdot P(12; 2, 2, 2, 2, 2, 1, 1)$.

(c). How many distinct arrangements of the letters in "PREPOSTEROUS" are there, in which the 5 vowels are consecutive? $P(5; 2, 2, 1)P(8; 2, 2, 2, 1, 1)$. Arrange the 5 vowels, and now think of them as one new letter V to be arranged with the remaining consonents.

(9). (20 points) Of a company's personnel, 7 work in design, 14 in manufacturing, 4 in testing, 5 in sales, 2 in accounting, and 3 in marketing. (Each of the 35 workers works in a single department.) A committee of 6 people is to be formed to meet with upper management. Answer each part separately.

(a). In how many ways can the committee be formed if there is to be exactly one member of each department? $7 \cdot 14 \cdot 4 \cdot 5 \cdot 2 \cdot 3$

(b). In how many ways can the committee be formed if there must be exactly two members from the manufacturing department? $\binom{14}{2}\binom{21}{4}$

Common mistakes: $\binom{14}{2}\binom{33}{4}$ allows more people from manufacturing.

(c). In how many ways can the committee be formed if there must be at least two members from the manufacturing department? $\binom{35}{6} - \binom{21}{6} - \binom{21}{5}\binom{14}{1}$.

Common mistake: double counting $\binom{14}{2}\binom{33}{4}$. Outcomes with more than 2 people from manufacturing will be counted several times. Remember the example from class, example 5(d) section 5.2.

(d). Lucy works in the design department, her husband Rickey works in marketing. In how many ways can the committee be formed if they cannot both be on the committee together? (It is ok for either Lucy or Rickey or neither of them to be on the committee.) $\binom{35}{6} - \binom{33}{4} = 2 \cdot \binom{33}{5} + \binom{33}{6}$. (Similar to example 5(e) section 5.2, done in class.)

Common mistakes: using wrong formula - for identical items. People are distinct! (Even people in the same department.)