AMS 301.2 (Spring, 2010) Estie Arkin

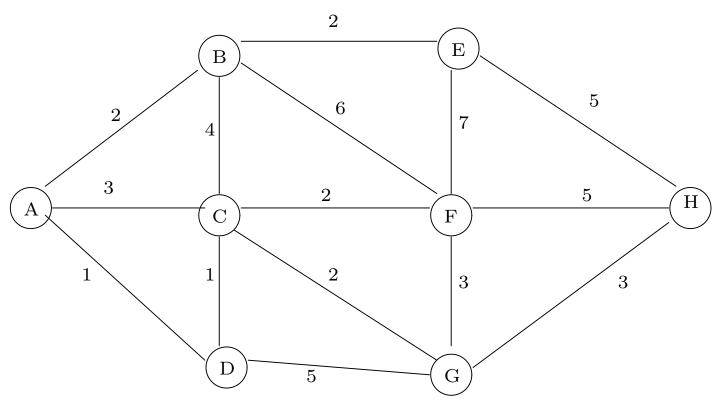
Exam 2 – Solution sketch

Mean 73.16, median 75, top quartile 82, high 95, low 36.

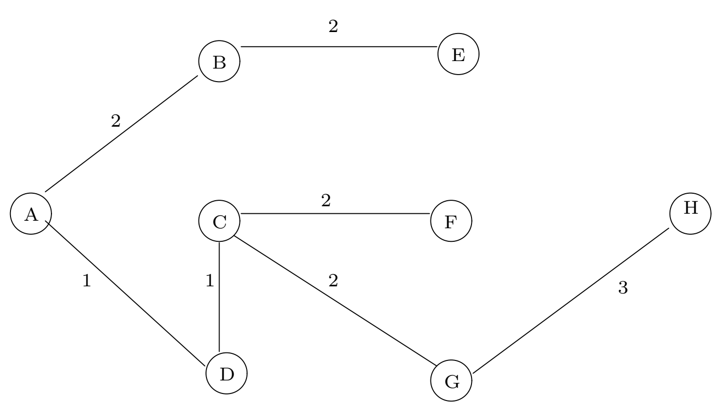
1. (5 points) Let T be a 4-ary tree with 200 internal nodes. How many leaves does the tree have? (A correct guess with no work shown will recieve very partial credit.)

$n = mi + 1$, $m = 4$, $i = 200$, so $n = 801$, $l = n - i = 601$

2. (9 points) Consider the following graph.



(a). Highlight the edges of a minimum spanning tree of the graph. Using Kruskal's algorithm, egdes would be put into the MST in the following order: (A,D) (C,D) (C,F) (B,A) (C,G) (B,E) (G,H). The cost of this tree is 13.



(b). Edge (A,B) is currently part of the MST, however its cost is uncertain. What are all the possibles costs of the edge for which it will be part of the MST? Explain briefly. (Your answer should be of the sort cost is greater than 3 and less than or equal to 17.)

Edge (A,B) connects the nodes B,E to the rest of the nodes A,C,D,F,G,H. Instead of it, we could use any one of the edges (B,C) (B,F) (E,F) (E,H), and we would choose the smallest cost among them, (B,C) of cost 4. So as long as the cost of (A,B) is at most 4, we would put it into the MST. If its cost is larger than 4, then instead we would put (B,C).

3. (20 points) True or False? Make sure to explain! (The best explanation if true, is a short proof,

and if false, a counterexample.) We are given a connected graph G with costs on edges. Assume all costs are positive and that there are no ties. A and B are two of the nodes in the graph.

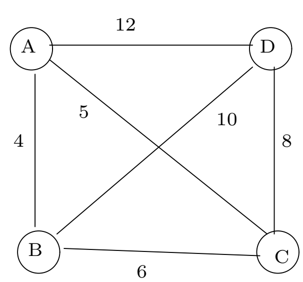
(a). A BFS on G rooted at node A has the same number of edges as a BFS on G rooted at node B. True. The number of edges in a spanning tree of G is always $n-1$ where n is the number of nodes.

(b). A BFS on G rooted at node A has the same number of leaves as a BFS on G rooted at node B. False. Consider the graph on nodes A,B,C,D with edges (A,B), (A,C), (A,D) and (B,C). A BFS starting at A would have 3 leaves B,C,D, whereas a BFS starting at B would have 2 leaves A,D.

(c). I calculate an MST using Prim's algorithm, starting with node A, and Tiffany calculates an MST using Prim's algorithm, starting with node B. We must obtain the same tree. True, since both yield an MST which is unique.

(d). Tiffany and I each calculate a quick TSP, using either of the algorithms discussed in class. We must obtain the same tour. False. Quick TSP may not give an optimal solution and so different calculations may result in different tours (of different costs). For example Consider K_4 as in the figure below. If we start a quick TSP with node A, we get $T_1 = A$, $T_2 = A, B, A$ $T_3 = A, C, B, A$ and $T_4 = A, D, C, B, A$. If instead we start at node B we get $T_1 = B$, $T_2 = B, A, B$ $T_3 = B, C, A, B$ and $T_4 = B, D, C, A, B$. If we use the "double the MST" method, we get an MST containing edges (A,B) (A,C) (C,D) and by shortcutting the Euler cycle (again there are several different options) we may get A, C, D, B, A.

(e). I calculate a the shortest path tree rooted at A, and Tiffany calculates the shortest path tree rooted at B. We must obtain the same shortest path tree. False. Consider the graph on three nodes A,B,C with edges (A, B) of length 2, edge (B,C) of length 3 and edge (A,C) of length 4. The shortest path tree rooted at A has edges (A,B) and (A,C), but the shortest path tree rooted at B has edges (A,B) and (B,C).



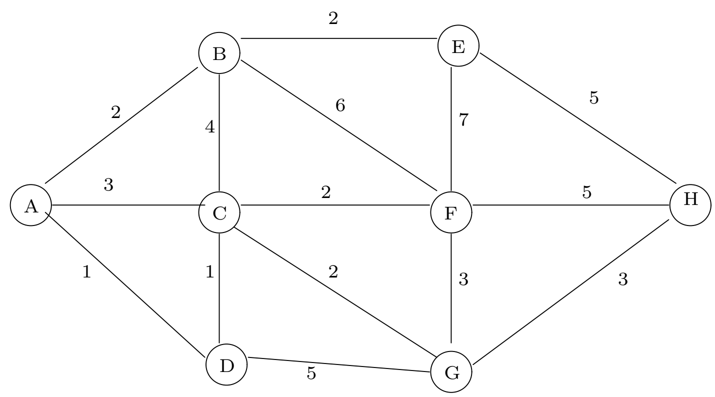
4. (6 points) We wish to model the word problem as a graph problem: The campus mail service must deliver interdepartmental mail to various buildings on campus. For simplicity, assume there is a single mailman who starts and ends his day at the central services building, and that he can carry all the mail to be delivered today in his cart at once.We also assume that we know how long it takes the mailman to walk between any two buildings.

(a). What do the nodes of the graph to be constructed represent? Buildings to which mail is to be delivered.

(b). What do the edges of the graph to be constructed represent? Possible travel between buildings.

(c). State which graph problem it is: Traveling Salesman Problem.

5. (10 points) (a). Highlight the edges of the shortest path tree rooted at node C for the graph below.

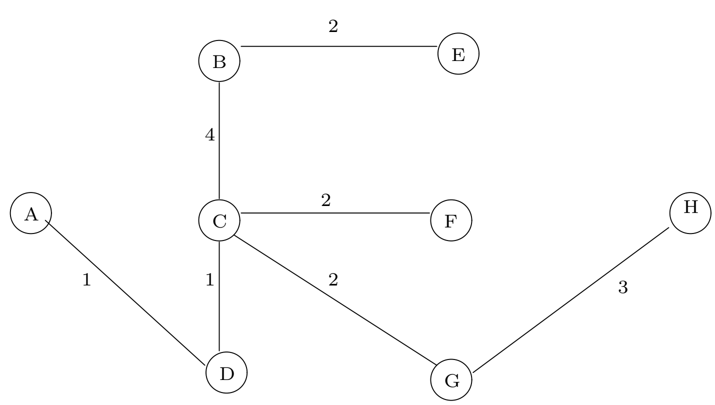


Dijsktra's algorithm calculates:

u_A	u_B	u_C	u_D	u_E	u_F	u_G	u_H	node becoming perm
3	4	0	1	∞	2	2	∞	(C and) node D
2	4	-	-	∞	2	2	∞	A
-	4	-	-	∞	2	2	∞	F
-	4	-	-	9	-	2	7	G
-	4	-	-	9	-	-	5	B
-	-	-	-	6	-	-	5	H
-	-	-	-	6	-	-	-	E

The final pred labels are: $\text{pred}_A = D$, $\text{pred}_B = C$, $\text{pred}_D = C$, $\text{pred}_E = B$ $\text{pred}_F = C$, $\text{pred}_G = C$, $\text{pred}_H = G$.

The shortest path tree rooted at C is:



(b). Edge (C,F) is currently part of the shortest path tree, however its length is uncertain. What are all the possibles lengthss of the edge for which it will be part of the shortest path tree rooted at C? Explain briefly. (Your answer should be of the sort cost is greater than 3 and less than or equal to 17.) If edge (C,F) does not exist, then the shortest path from C to F will be C-G-F of total length 5. So, as long as the cost of edge (C,F) is less than or equal to 5, it will a shortest path from C to F.

6. (8 points) We wish to arrange the digits 1,2,3,4,5,6,7,8,9.

(a). How many arrangements are possible? 9!

(b). How many arrangements have no adjacent odd digits? For example, the arrangement 385164927 is not allowed because 5 and 1 are adjacent. 5!4!. We arrange all the odd digits and all the even digits and then interleave them, odd, even, odd even ..., odd. Note that we must start and end with odd.

(7). (8 points) You throw four identical dice (each dice is 6 sided and shows the digits 1 through 6). Your write down the values showing in nondecreasing order from left to right. For example 2245 means you rolled two 2s, one 4 and one 5.

(a). How many outcomes are possible? $\binom{4+6-1}{4}$ Select 4 items out of 6 types, repeats allowed order not important (since we then write the outcomes in nondecreasing order).

(b). How many outcomes are possible in which all the values are different? $\binom{6}{2}$ since exactly 2 digits must be missing.

Common mistake: $6 \cdot 5 \cdot 4 \cdot 3 = P(6,4)$ which assumes that the order is important. Since the values are arranged in nondecreasing order after they are rolled, then it is as if order does not matter.

(8). (14 points) Two baseball teams A and B, play each other in a best of seven series (so that the first team to win 4 games wins the series). For example, the outcome ABAAA means that team B won the second game, team A won games 1,3,4,5 and therefore the series. Note that the series ends as soon as a team wins 4 games, and each game ends with one of the two teams winning (no ties).

(a). How many different outcomes are there if the series ends in exactly 6 games? $2\binom{5}{3} = 2P(5;3,2)$ there are 2 possible winners, A or B. The winner must win the 6th game and exactly 3 of the first 5 games.

Common mistake: $P(6;4,2)$ which says that a specific team won 4 games and the other team won 2 games. Since either team can be the winner of the series, we should multiply the result by 2. Furthermore, the winning team must win the last game, and so it must win 3 of the first 5 games, and the loosing team wins the remaining 2 games from the first 5 games.

(b). How many different outcomes are there? $2(\binom{6}{3} + \binom{5}{3} + \binom{4}{3} + \binom{3}{3})$

(9). (20 points) A 5 card hand is chosen at random from a standard deck of 52 cards. A standard deck has 13 cards from each of 4 suits (clubs, diamonds, hearts and spades).

(a). What is the probability that the hand contains a royal flush (Ace,King, Queen,Jack, 10 all in one suit)? $\frac{4}{\binom{52}{5}}$

(b). What is the probability that the hand contains exactly 3 jacks and exactly 2 clubs? Break into two cases depending on whether the jack of clubs is contained or not. If the jack of clubs is contained then we need to pick 2 more jacks from the remaining 3 jacks, 1 more club from the remaining 12 clubs and 1 more card that is neither a jack nor a club. If the jack of clubs is not contained, then we need to take the three remaining jacks, and 2 clubs from the 12 clubs (not including the jack).

$$\frac{\binom{3}{2}\binom{12}{1}\binom{52-13-3}{1}+\binom{3}{3}\binom{12}{2}}{\binom{52}{5}}$$

Common mistake: Considering a single case, or forgetting that in the first case an additional 5th card which is not a club nor spade must also be selected.

(c). What is the probability that the hand contains a pair (that is exactly two cards of the same denomination and three other cards that do not contain a pair, for example 2 jacks a four a queen and a nine)? Choose which card will be the "pair" 13 possibilities, then select 2 out of the 4 in that face value. Then select 3 more values out of the remaining 12, and select one from each such face value.

$$\frac{13\binom{4}{2}\binom{12}{3}4^3}{\binom{52}{5}}$$

Common mistake: Select the triple, and then multiply by $48 \cdot 44 \cdot 40$, this assumes the order of the other 3 cards is important, but it is not!