

**Exam 3 – Solution sketch**

Mean 75.89, median 80, top quartile 87, high 100 (3 of them!), low 18.

1. (20 points) Build a generating function for  $a_r$  in the following procedures. Remember to state which coefficient solves the initial problem. You do **not** need to calculate the coefficient.

(a). An exam has 20 questions worth 5 points each. Each problem receives 0,1,2,3,4,5 points. How many ways are there for a student to score  $r$  points? (Note that we do not care how many points any specific question receives, just the number of questions that receive 0 points, 1 point etc.)

Let  $e_1, \dots, e_{20}$  be the score on question 1-20, and we have  $e_1 + \dots + e_{20} = r$  where  $0 \leq e_i \leq 5$ . So we want the coef of  $x^r$  in  $g(x) = (1 + x + x^2 + x^3 + x^4 + x^5)^{20}$ .

(b). How many ways are there to distribute  $r$  identical forks to 10 people so that each person receives either one or two forks?

Coef of  $x^r$  in  $g(x) = (x + x^2)^{10}$ .

(c). The number of ways a team can score  $r$  points in a basketball game? (In basketball, any single shot is worth either one, two, or three points. We are only interested in the number of each type of shot made, not the order in which they were made.)

Let  $e_1, e_2, e_3$  be the number of shots that score 1,2,3 points. We have  $e_1 + 2e_2 + 3e_3 = r$  where  $0 \leq e_i$ . So we want the coef of  $x^r$  in  $g(x) = (1 + x + x^2 + x^3 + \dots)(1 + x^2 + x^4 + x^6 + \dots)(1 + x^3 + x^6 + x^9 + \dots)$ .

(d). In how many ways can we make change for  $r$  cents using 5 pennies, 3 nickels, and one dime?

Coef of  $x^r$  in  $g(x) = (1 + x + x^2 + x^3 + x^4 + x^5)(1 + x^5 + x^{10} + x^{15})(1 + x^{10})$ .

2. (5 points) Solve the following recurrence relation:  $a_n = 3a_{\frac{n}{3}} + 4$ ,  $a_1 = 1$

(You may assume that  $n = 3^m$ , for some  $m = 0, 1, 2, \dots$ )

Using the second formula from 7.2 we have  $c = k = 3$ ,  $d = 4$ ,

$$a_n = An - \frac{4}{3-1} = An - 2$$

Using the initial condition  $1 = a_1 = A - 2$  we get  $A = 3$ , so finally,  $a_n = 3n - 2$ .

3. (10 points) Consider the recurrence  $a_n = a_{n-1} + n$ ,  $a_1 = 1$ .

(a). Calculate  $a_2 = a_1 + 2 = 3$  and  $a_3 = a_2 + 3 = 6$ .

(b). Solve the recurrence relation. Make sure to verify your answer using induction.

We “guess”  $a_n = n(n+1)/2$  and verify by induction:  $a_1 = 1 \cdot 2/2 = 1$ .

IHOP:  $a_n = n(n+1)/2$ , we need to show  $a_{n+1} = (n+1)(n+2)/2$ .

Proof:  $a_{n+1} = a_n + (n+1) = n(n+1)/2 + n+1 = (n^2 + n + 2n + 2)/2 = (n+1)(n+2)/2$ .

Alternatively, we could say that  $a_n = 1 + 2 + \dots + n$  and prove this formula.

4. (15 points) (a). What is the coefficient of  $x^{50}$  in the expansion of  $(x^9 + x^{10} + x^{11} + \dots)^3$ ?

Same as coefficient of  $x^{50-27} = x^{23}$  in  $(1 + x + x^2 + x^3 + \dots)^3$ . Using formula (5) of section 6.2,  $\binom{23+3-1}{23}$ .

(b). What is the coefficient of  $x^{14}$  in the expansion of  $\frac{1+x+x^4}{(1-x)^5}$ ?

For  $(1-x)^{-5}$  by formula (5) of section 6.2, the coefficients are  $b_r = \binom{r+5-1}{r}$ .

Using formula (6), we combine and get:  $1\binom{14+5-1}{14} + 1\binom{13+5-1}{13} + 1\binom{10+5-1}{10}$

5. (10 points) A produce stand sells only broccoli, carrots and okra. One day the stand served 207 customers. 114 people purchased broccoli, 152 purchased carrots, 25 purchased okra, 64 purchased broccoli and carrots, 12 purchased carrots and okra and 9 purchased all 3. How many people purchased broccoli and okra?

Define  $A$  the people that purchased broccoli,  $B$  the purchased carrots, and  $C$  the people that purchased okra. So  $N(A \cup B \cup C) = 207$ ,  $N(A) = 114$ ,  $N(B) = 152$ ,  $N(C) = 25$ ,  $N(A \cap B) = 64$ ,  $N(B \cap C) = 12$ ,  $N(A \cap B \cap C) = 9$ . We want to find  $N(A \cap C)$ .

$N(A \cup B \cup C) = 207 = N(A) + N(B) + N(C) - N(A \cap B) - N(A \cap C) - N(B \cap C) + N(A \cap B \cap C) = 114 + 152 + 25 - 64 - 12 - N(A \cap C) + 9$ , so  $N(A \cap C) = 17$ .

6. (20 points) A fast food outlet gives away 4 different toys in children's meal packs, one toy per pack. If we buy 10 children's meal packs, what is the probability of getting all 4 toys?

The problem does not specify that each meal gets one of the 4 toys with equal probability. Under this assumption, the correct way to solve the problem is similar to problem 17 section 8.1 done in class, where we assign distinct objects (people) to distinct rooms. Here we are assigning distinct books to students. Define  $A_i$  the event that  $i$ th toy is not gotten, and use inclusion exclusion to calculate the numerator.

$$N(\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 \cap \bar{A}_4) = 4^{10} - 4 \cdot 3^{10} + \binom{4}{2} 2^{10} - 4 + 0$$

To get the probability we divide by  $4^{10}$ .

Since the problem statement is somewhat unclear, I also accepted (incorrect) answers using formula for identical objects  $\binom{r+n-1}{r}$ . This answer is not correct, since the outcomes counted in the numerator would no longer have equal probability.

7. (20 points) (a). Write a recurrence relation for  $a_n$  the number of  $n$  digit binary sequences with at least one instance of consecutive 0s. (You do not have to solve the recurrence.)  $a_n = a_{n-1} + a_{n-2} + 2^{n-2}$  the first term is when the first digit is 1, then we need a binary sequence of length  $n-1$  that contains consecutive zeros. The second term is when the first digit is 0 and then the second digit is 1. Now we need a sequence of  $n-2$  digits containing consecutive 0's. The third term is when the first and second digits are both 0, in which case any binary sequence with  $n-2$  digits is allowed.

(b). Write a complete set of initial conditions for your recurrence in part (a).

$$a_1 = 0 \quad a_2 = 1$$

(c). Calculate  $a_3$  and  $a_4$  using your recursion from (a) and initial conditions from (b).

$$a_3 = 3, \quad a_4 = 8.$$

(d). Write a recurrence relation for  $a_n$  the number of  $n$  digit ternary (0,1,2) sequences with at least one instance of consecutive 0s. (You do not have to solve the recurrence.)

$$a_n = 2a_{n-1} + 2a_{n-2} + 3^{n-2}$$

(e). Write a complete set of initial conditions for your recurrence in part (d).

$$a_1 = 0 \quad a_2 = 1$$