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## Two Theorems on Hamilton circuits:

The following theorems may be useful in proving that a graph does not have a Hamilton circuit. (See Theorem 1, second edition of Applied Combinatorics.)

**Theorem 0:** *If a graph $G$ has a set $S$ of $|S| = k$ vertices whose removal from $G$ results in a graph $G \setminus S$ which has more than $k$ connected components, then $G$ does not have a Hamilton circuit.*

**Proof:** We prove the theorem by way of contradiction. Suppose $G$ has a Hamilton circuit. When leaving nodes of a connected component of $G \setminus S$, a Hamilton circuit can go only to $S$. Each "arrival" in $S$ must be at a differect vertex of $S$. Hence, $S$ must have at least as many vertices as $G \setminus S$ has connected components, contrary to the statement given in the theorem. □

**Example 1:** Figure 2.4 Section 2.2: Removing one node $c$ results in two connected components $\{a, d\}$, $\{b, e\}$.

**Example 2:** Problem 4(a) Section 2.2: Removing two nodes $a, e$ results in three connected components $\{b\}$, $\{c\}$, $\{d\}$.

**Example 3:** Problem 7(a) Section 2.2: Removing three nodes $e, i, g$ results in five connected components $\{a, b, c\}$, $\{f\}$, $\{h\}$, $\{j\}$, $\{d\}$.

Note that Theorem 0 is not an "if and only if" theorem. For some graphs, there is no such subset $S$ of nodes, but yet a Hamilton circuit does not exist. For example the graph in Problem 4(n) Section 2.2, removal of any set $S$ of nodes does not result in more than $|S|$ connected components, yet $G$ does not have a Hamilton circuit.

A special case of Theorem 0 is the following theorem for bipartite graphs: (See problem 9 Section 2.2.)

**Theorem 5:** *If a bipartite graph $G$, with nodes $R$ on the right, and nodes $L$ on the left, has $|R| \neq |L|$, then $G$ does not have a Hamilton circuit.*

**Proof:** Suppose without loss of generality that $|R| > |L|$. Then removing the nodes of $L$ from $G$ results in $|R|$ connected components, so by Theorem 0, $G$ does not have a Hamilton circuit. Alternatively: A Hamilton circuit in a bipartite graph must alternate betweeen Right and Left vertices, and therefore must have the same number of Right and Left vertices. □

**Example 4:** Problem 4(p) Section 2.2: The graph is bipartite with $R = \{h, d, b, c, i, j\}$ and $L = \{a, f, e, g, k\}$. Since $R$ has 6 vertices and $L$ has 5, the graph does not have a Hamilton circuit.

Note that if a bipartite graph has an odd number of nodes, then the graph has $|R| \neq |L|$, and therefore does not have a Hamilton circuit.

Important! This theorem is not helpful if the graph $G$ is not bipartite. For instance $K_5$ is not bipartite, but has a Hamilton circuit. Also, if $G$ is bipartite and $|R| = |L|$, we do not know whether $G$ has a Hamilton circuit.