Remarks About Methods of Proof

We illustrate some typical proof methods below. For this, we will use the following two definitions:

Def. An integer n is *even* if it can be written as $n = 2k$ for some integer k.

Def. An integer n is *odd* if it can be written as $n = 2k + 1$ for some integer k.

1. DIRECT PROOF: Start with the premise of the theorem and draw conclusions from it until you arrive at the desired conclusion.

Theorem 1 *If n is an even integer then n^2 is even.*

Proof. We know that there exists an integer k such that $n = 2k$. Therefore

$$n^2 = (2k)^2 = 4k^2 = 2(2k^2).$$

Note that since k is an integer, so is $(2k^2)$, and so we expressed n^2 as 2 times an integer, showing that n^2 is even. □

2. PROVING THE CONTRAPOSITIVE: In this method of proof we want to show that **if** (statement A) **then** (statement B). Instead we show an equivalent fact: **if** (not statement B) **then** (not statement A).

Theorem 2 *Let n be an integer. If n^2 is even then n is even.*

Proof. In this case statement A is "n^2 is even", and statement B is "n is even". We show instead that if n is odd (not statement B) then n^2 is odd (not A). We know there exists an integer k such that $n = 2k + 1$, therefore

$$n^2 = (2k + 1)^2 = 4k^2 + 2k + 1 = 2(2k^2 + k) + 1$$

Since k is integer, so is $2k^2 + k$ and therefore we have shown that n^2 is odd. □

3. PROOF BY CONTRADICTION: In this method of proof we want to show that **if** (statement A) **then** (statement B). We assume that (statement A) is true **and** (not statement B) is true, and look for a contradiction. Once we get a contradiction, we conclude that our assumption was false, and therefore the theorme is true!

Theorem 3 *Let n and m be integers. If $n \cdot m$ is even then at least one of n or m must be even.*

Proof. Assume $n \cdot m$ is even (statement A) and neither n nor m are even (not statement B). Therefore we can write: $n = 2k + 1$ and $m = 2c + 1$ for some integers k, c.

$$n \cdot m = (2k + 1)(2c + 1) = 4k \cdot c + 2k + 2c + 1 = 2(2k \cdot c + k + c) + 1$$

showing that $n \cdot m$ is odd. Since this is a contraditction, we conclude that the theorme is true. □

4. PROOF BY INDUCTION: We wish to show that a statement $S(n)$ is true for all integer $n \geq n_0$ (for some number n_0). To do so we can show two things:

1. Base case: $S(n_0)$ is true.
2. Inductive hypothesis (IH): Assume that $S(n)$ is true for some $n \geq n_0$ and show that $S(n+1)$ is true.

Theorem 4 *For any $n \geq 0$ and $x \neq 1$,*

$$1 + x + x^2 + \dots x^n = \frac{x^{n+1} - 1}{x - 1}$$

Proof. We first show the base case is true. For $n = 0$ the statement $S(0)$ claims that

$$1 = \frac{x^{0+1} - 1}{x - 1},$$

which is clearly true.

Now, we make the following induction hypothesis (IH):

$$1 + x + x^2 + \dots x^n = \frac{x^{n+1} - 1}{x - 1}.$$

We need to show that

$$1 + x + x^2 + \dots + x^n + x^{n+1} = \frac{x^{n+2} - 1}{x - 1}.$$

$$\begin{aligned} 1 + x + x^2 + \dots + x^n + x^{n+1} &= \frac{x^{n+1} - 1}{x - 1} + x^{n+1} \\ &= \frac{x^{n+1} - 1 + (x-1)x^{n+1}}{x - 1} \\ &= \frac{x^{n+1} - 1 + x^{n+2} - x^{n+1}}{x - 1} \\ &= \frac{x^{n+2} - 1}{x - 1} \end{aligned}$$

The first equality follows from the inductive hypothesis (IH), and the rest is simple algebra. □

Remark: It is usually trivial to show that the base case is indeed true; however, it is still very important to do so! For example, consider the following "proof" to an incorrect theorem:

Theorem 5 (WRONG) *For any integer $n \geq 0$ $n = n + 5$.*

Proof. "Proof" by induction: Assume the fact is true for some n, namely $n = n + 5$, we need to show it is true for $n + 1$, namely $(n + 1) = (n + 5) + 1$. This is "easy" to show: Simply take the (IH) and add 1 to each side of the equality. □

The problem with the alleged "proof" is that we did not check that the base case is true, and clearly, in this case, it is NOT true.

Theorem 6 (WRONG) *In any set of n students, all students have the same height.*

Proof. "Proof" by induction on the number of students in a set: We start with the "base" of the induction: For any set of size 1, the claim is clearly true (that all students in that set have the same height). Thus, let us assume that the claim is true for sets of size k (this is the induction hypothesis, IH). Now consider a set, S, of size $k + 1$. Then, $S = S' \cup \{p_{k+1}\}$, where $S' = \{p_1, \dots, p_k\}$ is a set of size k, and p_i denotes "person $\#i$." By the IH, all students in S' have the same height; in particular, p_1 has the same height as p_2. But also, $S = \{p_1\} \cup S''$, where $S'' = \{p_2, \dots, p_{k+1}\}$ is a set of size k. So all students in S'' have the same height; in particular, p_{k+1} has the same height as p_2, which we already know has the same height as p_1 (and all other members of S'). Thus, all $k + 1$ students in S have the same height, and we are done, by induction. *What is wrong with this "proof"?* □

5. DISPROOF BY COUNTEREXAMPLE: To show that a "theorem" is false, one needs to present a counterexample. For instance, for (WRONG) Theorem 5, we can disprove the claim simply by presenting the case of $n = 4$, and noting that the claim fails for this choice of n ($4 \neq 4 + 5$). As another example, consider

Theorem 7 (WRONG) *For any n, the number $3n$ is even.*

A counterexample is the case $n = 7$, since $3 \cdot 7 = 21$ is not even. (The claim does hold for some numbers n; e.g., $n = 4$.)

Note that a "theorem" may make a claim that is in fact true for many instances (even an infinite number of instances); however, if there is even one counterexample, then the "theorem" is not a theorem.

6. TWO *WRONG* METHODS OF PROOF: We conclude with two incorrect methods to "prove" a theorem:

1. (*"Proof" by example*) In this case the statement is shown to be true for one or more (but not all) examples. For instance, we can attempt to prove (WRONG) Theorem 7 by showing that for $n = 4$ (and, in fact, for any even number n) the claim does hold. This does NOT mean that it holds for every n.

2. (*"Proof" by lack of counterexample*) In this case we try to claim that a statement is true simply because we could not find an instance for which it is false! Unfortunately, all we can really conclude is that a counterexample may be difficult to find (or that we were too blind to see it) – not that it does not exist.