1. (9 points) Consider the following LP:

\[
\begin{align*}
\text{min} & \quad z = -2x_1 + x_3 \\
\text{s.t.} & \quad -x_1 + 5x_2 + x_3 \geq 1 \\
& \quad x_1 - 2x_2 + x_3 = 10 \\
& \quad x_1, x_3 \geq 0 \\
& \quad x_2 \text{ unrestricted}
\end{align*}
\]

Rewrite the LP in standard form.

\[
\begin{align*}
\text{min} & \quad z = -2x_1 + x_3 \\
\text{s.t.} & \quad -x_1 + 5x_2' - 5x_2'' + x_3 - e_1 = 1 \\
& \quad x_1 - 2x_2' + 2x_2'' + x_3 = 10 \\
& \quad x_1, x_3, x_2', x_2'' \geq 0
\end{align*}
\]

2. (15 points) Consider the feasible region given by the following constraints: (It may be helpful to sketch it and/or put it into standard form.)

\[
\begin{align*}
x_1 + x_2 & \leq 6 \quad (1) \\
x_1 & \leq 2 \quad (2) \\
x_2 & \leq 4 \quad (3) \\
x_1 & \geq 0 \quad (4) \\
x_2 & \geq 0 \quad (5)
\end{align*}
\]

In standard form, there are 3 constraints (and the non negativity constraints) and 5 variables, so there should be 3 basic variables and 2 non basic variables.

\[
\begin{align*}
x_1 + x_2 + s_1 & = 6 \quad (6) \\
x_1 + s_2 & = 2 \quad (7) \\
x_2 + s_3 & = 4 \quad (8) \\
x_1, x_2, s_1, s_2, s_3 & \geq 0 \quad (9)
\end{align*}
\]

(a). Is the point \(x_1 = 0, x_2 = 4\) a feasible point? Is it a basic solution? Yes, it is feasible (constraints are satisfied) and basic, \(x_2, s_1, s_2\) are basic, \(x_1, s_3\) non basic.

(b). Is the point \(x_1 = 2, x_2 = 2\) a feasible point? Is it a basic solution? Yes, it is feasible (constraints are satisfied) but not basic, since \(x_1, x_2, s_1, s_3\) are all positive, so we would need to have 4 basic variables, but there are only 3 constraints.
(c). The point \( x_1 = 2, x_2 = 4 \) is a basic feasible solution. Is it a degenerate basic feasible solution? Yes it is feasible and is a degenerate BFS, \( x_1, x_2 \) are basic, and all slack variables are equal to zero, one of them must be basic, and therefore it is degenerate. (Basic variable equals to zero.)

3. (15 points) You are given the tableau for a max problem. Give conditions on the unknowns \( a_1, a_2, a_3, b, c \) that make the following true. Your conditions should be as general as possible (don’t just give an example, such as \( a_1 = 3 \)).

<table>
<thead>
<tr>
<th>( z )</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( x_4 )</th>
<th>( x_5 )</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>c</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>10</td>
</tr>
<tr>
<td>0</td>
<td>-1</td>
<td>( a_1 )</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>0</td>
<td>( a_2 )</td>
<td>-4</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>( a_3 )</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>( b )</td>
</tr>
</tbody>
</table>

(a). The current BFS is optimal. To be feasible, \( b \geq 0 \), for optimality, \( c \geq 0 \). \( a_1, a_2, a_3 \) can be anything.

(b). The current BFS is optimal and there are multiple optimal solutions. To be feasible, \( b \geq 0 \), for multiple optimal solutions, \( c = 0 \). \( a_1, a_2, a_3 \) can be anything.

(c). The LP is unbounded. To be feasible, \( b \geq 0 \), for unbounded objective, \( c < 0 \). \( a_1 \) can be anything, \( a_2, a_3 \leq 0 \).

4. (8 points) Consider the following LP:

\[
\begin{align*}
\text{min } & \quad z = 3x_1 + x_2 \\
\text{s.t. } & \quad x_1 - x_2 + x_3 = 1 \\
& \quad x_2 + x_4 = 2 \\
& \quad x_1 + x_2 - x_5 = 6 \\
& \quad x_1, x_2, x_3, x_4, x_5 \geq 0
\end{align*}
\]

We wish to solve this problem using the big M method. Set up the first tableau (in canonical form!)
we should use.

The problem is in standard form, and \( x_3, x_4 \) can be basic variables for the first and second constraints, so we add an artificial variable to the third constraint and get:

<table>
<thead>
<tr>
<th>( z )</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( x_4 )</th>
<th>( x_5 )</th>
<th>( a_3 )</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-3</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>(-M)</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>6</td>
</tr>
</tbody>
</table>

We then “clean-up” the objective row so that basic variable \( a_3 \) has objective coefficients zero, so the tableau is in canonical form:

<table>
<thead>
<tr>
<th>( z )</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( x_4 )</th>
<th>( x_5 )</th>
<th>( a_3 )</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-3+M</td>
<td>-1+M</td>
<td>0</td>
<td>0</td>
<td>-M</td>
<td>0</td>
<td>6M</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>6</td>
</tr>
</tbody>
</table>
5. (27 points) A bank is open Monday-Friday from 9am to 5pm. From past experience, the bank knows that it needs (at least) the following number of tellers:

<table>
<thead>
<tr>
<th>Time period</th>
<th>9-10</th>
<th>10-11</th>
<th>11-noon</th>
<th>noon-1</th>
<th>1-2</th>
<th>2-3</th>
<th>3-4</th>
<th>4-5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tellers required</td>
<td>4</td>
<td>3</td>
<td>4</td>
<td>6</td>
<td>5</td>
<td>6</td>
<td>8</td>
<td>8</td>
</tr>
</tbody>
</table>

The bank hires two types of tellers. Full time tellers work 9-5 every day, except for 1 hour off for lunch. (The bank determines when a full time teller takes lunch hour, but it must be either noon-1 or 1-2.) Full time employees are paid $8 per hour (this includes payment for the lunch hour). The bank can also hire part time tellers. Each part time teller must work exactly 3 consecutive hours each day, and gets paid $5 per hour. To maintain quality of service, at most 5 part time tellers can be hired. Formulate an LP to minimize the cost of the bank to meet teller requirements. (Your formulation does NOT have to be put into standard form.)

(a). Define the variables you are using in the formulation. Let \( x_{i2} \), \( x_1 \) be the number of full time tellers that take their lunch starting at 12 or 1 respectively. Let \( y_i \) be the number of part time tellers that start their 3 hour shift at \( i = 9, 10, 11, 12, 1, 2 \). Note that full time tellers get paid 64$ (8$ for each of 8 hours) and part time tellers get paid 15$. Also, since part time tellers work exactly 3 consecutive hours, I did not use variables \( y_3, y_4 \).

(b),(c). The objective function and the constraints are:

\[
\begin{align*}
\text{min } & \quad z = 64(x_{12} + x_1) + 15(y_9 + y_{10} + y_{12} + y_1 + y_2) \\
& \quad x_{12} + x_1 + y_9 \geq 4 \\
& \quad x_{12} + x_1 + y_9 + y_{10} \geq 3 \\
& \quad x_{12} + x_1 + y_9 + y_{10} + y_{11} \geq 4 \\
& \quad x_1 + y_{10} + y_{11} + y_{12} \geq 6 \\
& \quad x_{12} + y_{11} + y_{12} + y_1 \geq 5 \\
& \quad x_{12} + x_1 + y_{12} + y_1 + y_2 \geq 6 \\
& \quad x_{12} + x_1 + y_1 + y_2 \geq 8 \\
& \quad x_{12} + x_1 + y_2 \geq 8 \\
& \quad y_9 + y_{10} + y_{11} + y_{12} + y_1 + y_2 \leq 5 \\
& \quad x_{12}, x_1, y_9, y_{10}, y_{11}, y_{12}, y_1, y_2 \geq 0
\end{align*}
\]

Common mistakes: Forgetting lunch for the full time workers. Defining a variable \( x_i \) how many full (or part) time workers are working at hour \( i \). Full time workers must work all hours except lunch! And part time workers must work exactly 3 consecutive hours.

6. (16 points) A company produces and sells wooden soldiers and wooden trains. Each soldier requires 3 board feet of lumber and 2 hours of labor. Each train requires 5 board feet of lumber and 4 hours of labor. A total of 145 board feet of lumber and 90 hours of labor are available. Up to 50 soldiers and 50 trains can be sold. Trains sell for $55, and soldiers for $32. In addition to producing trains and soldiers itself, the company can buy (from an outside supplier) extra soldiers at $27 each and extra trains at $50 each. Let \( SM, TM \) be the number of soldiers and trains made by the company, and \( SB, TB \) the number of soldiers and trains bought from the supplier. Use the Lindo output below to answer each of the following parts.
\[
\begin{align*}
\text{max} & \quad 32SM + 55TM + 5SB + 5TB \\
\text{s.t.} & \quad 2) \quad 3SM + 5TM \leq 145 \\
& \quad 3) \quad 2SM + 4TM \leq 90 \\
& \quad 4) \quad SM + SB \leq 50 \\
& \quad 5) \quad TM + TB \leq 50 \\
\text{objective function value} & \quad 1715.0000 \\
\text{variable} & \quad \text{value} & \quad \text{reduced cost} \\
SM & \quad 45.00000 & \quad 0.00000 \\
TM & \quad 0.00000 & \quad 4.000000 \\
SB & \quad 5.000000 & \quad 0.000000 \\
TB & \quad 50.000000 & \quad 0.000000 \\
\text{row} & \quad \text{slack or surplus} & \quad \text{dual prices} \\
2) & \quad 10.000000 & \quad 0.000000 \\
3) & \quad 0.000000 & \quad 13.500000 \\
4) & \quad 0.000000 & \quad 5.000000 \\
5) & \quad 0.000000 & \quad 5.000000 \\
\text{Range in which basis remains unchanged:} \\
\text{OBJ coefficient ranges} \\
\text{variable} & \quad \text{current coef} & \quad \text{allowable increase} & \quad \text{allowable decrease} \\
SM & \quad 32.000000 & \quad \infty & \quad 2.000000 \\
TM & \quad 55.000000 & \quad 4.000000 & \quad \infty \\
SB & \quad 5.000000 & \quad 2.000000 & \quad 5.000000 \\
TB & \quad 5.000000 & \quad \infty & \quad 4.000000 \\
\text{righthand side ranges} \\
\text{row} & \quad \text{current RHS} & \quad \text{allowable increase} & \quad \text{allowable decrease} \\
2) & \quad 145.000000 & \quad \infty & \quad 10.000000 \\
3) & \quad 90.000000 & \quad 6.66667 & \quad 90.000000 \\
4) & \quad 50.000000 & \quad \infty & \quad 5.000000 \\
5) & \quad 50.000000 & \quad \infty & \quad 50.000000 \\
\end{align*}
\]

(a). If the company can purchase trains for $48, what would be the new optimal profit? The profit on TB increases from 5 to 7, which is within the allowable range (infinity). The new \( z = 1715 + 2 \cdot 50 = 1815 \).

Common mistake: Confusing the purchase price of trains vs the selling price (the coefficient of TM). Or, calling this a decrease. It is an increase in the coefficient of TB from 5 to 7.

(b). What is the most that the company should be willing to pay to for another board foot of lumber? The dual price of lumber, which is zero.

(c). If only 40 trains could be sold, what would be the new optimal solution (the \( z \))? Decreasing the RHS in row 5 (4th constraint) from 50 to 40, is within range (the allowable decrease is 50). The new \( z = 1715 + (-10)5 = 1665 \).

(d). If only 40 trains could be sold, and 91 hours of labor are available, what would be the new optimal solution (the \( z \))? Using the 100% rule, we first check the range: \( \frac{50-40}{90} + \frac{91-90}{6.6667} < 1 \). The new \( z = 1715 + (-10)5 + 1(13.5) = 1678.5 \)

7. (10 points) The tableau below is for Phase I of the Two Phase Method. \( a_1 \) and \( a_2 \) are the
artificial variables or constraints 1,2, $e_1, e_2$ are the excess variables of constraints 1 and 2, and $s_3$ is the slack variable of the third constraint.

$$
\begin{array}{cccccccc}
  w & x_1 & x_2 & e_1 & e_2 & s_3 & a_1 & a_2 & \text{RHS} \\
  1 & 0 & 0 & -1/2 & -1 & -1/2 & -1/2 & 0 & 1/2 \\
  0 & 1 & 0 & -3/4 & 0 & -1/4 & 1 & 0 & 9/4 \\
  0 & 0 & 0 & -1/2 & -1 & -1/2 & 1/2 & 1 & 1/2 \\
  0 & 0 & 1 & 1/2 & 0 & 1/2 & -1/2 & 0 & 2 \\
\end{array}
$$

(a). At this tableau, we have:

\[
w = 1/2
\]

\[
x_1 = 9/4
\]

\[
x_2 = 2
\]

\[
e_1 = 0
\]

\[
e_2 = 0
\]

\[
s_3 = 0
\]

\[
a_1 = 0
\]

\[
a_2 = 1/2
\]

(b). The basic variables for this tableau are: $w, x_1, x_2, a_2$.

(c). The tableau shows an optimal solution to the Phase I LP. Is the original LP feasible? Explain briefly. Since $w = a_2 = 1/2 > 0$ the original LP is not feasible.