AMS 540 / MBA 540 (Fall, 2006) Estie Arkin

Linear Programming - Final solution sketch

Mean 62.36, median 75, high 100, low 15.

1). (10 points) We wish to solve an integer programming problem. All variables are restricted to be integer. We began by solving the LP relaxation of the problem and got the final (optimal) tableau for it. Unfortunately, not all the variables are integer.

z	x_1	x_2	x_3	s_1	s_2	s_3	RHS
1	0	-2	-3	-1.5	0	0	3.5
0	1	0	1.5	-2.2	0	0	4.2
0	0	2	0	-1.7	0	1	12
0	0	2.4	-3.3	0	1	0	6

To solve the problem using the cutting plane method, what cut (constraint) would you add? (You do not have to solve the new LP, just find the cut.)

Using the first constraint row of the tableau, we get: $x_1 + 1.5x_3 - 2.2s_1 = 4.2$, which we rewrite as: $x_1 + x_3 - 3s_1 - 3 = 0.2 - 0.5x_3 - 0.8s_1$. So the cut is: $0.2 - 0.5x_3 - 0.8s_1 \le 0$

Common mistake: using the last row, in which the rhs is integer. This cannot produce a cut!

2). (20 points) A company must meet the following demands for cash at the beginning of each of the next six months: Month 1 - \$200; month 2 - \$100; month 3 - \$50; month 4 - \$80; month 5 - \$160; month 6 - \$140. At the beginning of month 1, the company has \$150 in cash and \$200 worth of bond 1, \$100 worth of bond 2, and \$400 worth of bond 3. Of course the company will have to sell some bonds to meet demands, but a penalty will be charged for any bonds sold before the end of month 6. The penalties for selling \$1 worth of each bond are given in the table below:

Bond	Month 1	Month 2	Month 3	Month 4	Month 5	Month 6
1	\$0.21	\$0.19	\$0.17	\$0.13	\$ 0.09	\$0.05
2	\$0.50	\$0.50	\$0.50	\$0.33	\$ 0	\$0
3	\$1.00	\$1.00	\$1.00	\$1.00	\$ 1.00	\$0

(a). Assuming that all bills must be paid on time, formulate a balanced transportation problem that can be used to minimize the cost of meeting the cash demands for the next six months. Represent your formulation as a transportation tableau (cost and requirement table).

	Month 1	Month 2	Month 3	Month 4	Month 5	Month 6	dummy	supply
Cash	\$0	\$0	\$0	\$0	\$ 0	\$0	\$0	150
Bond 1	\$0.21	\$0.19	\$0.17	\$0.13	\$ 0.09	\$0.05	\$0	200
Bond 2	\$0.50	\$0.50	\$0.50	\$0.33	\$ 0	\$0	\$0	140
Bond 3	\$1.00	\$1.00	\$1.00	\$1.00	\$ 1.00	\$0	\$0	400
Demand	200	100	50	80	160	140	120	850

(b). Assume that the payment of bills can be made after they are due, but a penalty of \$0.05 per month is assessed for each dollar of cash demand that is postponed for one month. Assume that all bills must be paid by the end of month 6. Formulate a transshipment problem that can be used to minimize the cost of paying the next 6 months bills.

We construct the following graph, with three types of nodes: The first type are the suppliers Cash, B1,B2,B3, each with supply as given. The third type of nodes are the demand nodes, one for each month, and a dummy, with their demand as in part (a). The second type nodes are transshipment nodes, they will correspond to selling at a particular point in time. There are 7 such nodes, for the beginning of months 1,...,7. Edges: From supply nodes to transhipment nodes, with cost being the penalty. From supply nodes to dummy at cost zero. From cash node to all demand nodes, at zero cost. From transshipment nodes to demand nodes the cost corresponds to the penalty: From transshipment node i to demand j the cost is zero if $i \le j$ and the cost is $0.05(i - j)$ if $i > j$.

Another formulation: Define only the nodes Cash, B1,B2,B3 and months 1-6, have additional arcs from month i to month $i-1$ of cost 0.05. Other arcs from supply nodes to demand nodes are as in part (a), with the same costs.

3). (30 points) The following LP was solved (using the big M method) and the optimal tablaeu is given below. e_1 and e_2 are the excess variables subtracted from the first and second constraints, and a_i is the artificial variable of the ith constraint.

$$\begin{aligned} \max \quad & z = 4x_1 + x_2 \\ \text{s.t.} \quad & 3x_1 + x_2 \geq 6 \\ & 2x_1 + x_2 \geq 4 \\ & x_1 + x_2 = 3 \\ & x_1, x_2 \geq 0 \end{aligned}$$

z	x_1	x_2	e_1	e_2	a_1	a_2	a_3	rhs
1	0	3	0	0	M	M	$M+4$	12
0	1	1	0	0	0	0	1	3
0	0	2	1	0	-1	0	3	3
0	0	1	0	1	0	-1	2	2

(a). Find the dual of this LP and its optimal solution (the objective value and the value of the dual variables). Use the tableau - do not solve from scratch!

$$\begin{aligned} \min \quad & 6w_1 + 4w_2 + 3w_3 \\ \text{s.t.} \quad & 3w_1 + 2w_2 + w_3 \geq 4 \\ & w_1 + w_2 + w_3 \geq 1 \\ & w_1, w_2 \leq 0, \;\; w_3 \text{ unrestricted} \end{aligned}$$

The optimal dual solution is $w_1 = w_2 = 0$, $w_3 = 4$, and the objective function is 12.

(b). Find the range of values of the objective function coefficient for x_1 for which the current basis remains optimal. The new row zero is given by the old row zero $+\Delta$ times the row for which x_1 is basic. We can ignore the artificial variables, so we need only check for x_2, and we have new$z_2 - c_2 = 3 + \Delta \cdot 1 \geq 0$ so we need $\Delta \geq -3$. The range is therefore $c_1 \geq 4 - 3 = 1$.

(c). Find the range of values of b_2 for which the current basis remains optimal. Compute the new rhs as $B^{-1}b'$.

$$\begin{bmatrix} 0 & 0 & 1 \\ -1 & 0 & 3 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ 4+\Delta \\ 3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 2-\Delta \end{bmatrix}$$

so $\Delta \leq 2$ or $b_2 \leq 6$.

(d). We wish to add to the LP the constraint $x_1 \leq 2$, for which the current optimal solution is not feasible. Set up a tableau on which to proceed by the dual Simplex method to find the new optimal solution.

We can ignore the artificial variables here. Add a new row with new slack variable and clean up:

z	x_1	x_2	e_1	e_2	s_4	rhs
1	0	3	0	0	0	12
0	1	1	0	0	0	3
0	0	2	1	0	0	3
0	0	1	0	1	0	2
0	0	-1	0	0	1	-1

(e). Solve the problem set up in part (d) using the dual simplex method. Note, if you are doing more than 2 pivots, something is wrong!

After 1 pivot we get a new optimal tableau:

z	x_1	x_2	e_1	e_2	s_4	rhs
1	0	0	0	0	3	9
0	1	0	0	0	1	2
0	0	0	1	0	2	1
0	0	0	0	1	1	1
0	0	1	0	0	-1	1

4). (10 points) Consider a Transshipment problem, and supposed we are given a feasible solution to it. Describe an algorithm to turn it into a feasible tree solution. Your algorithm should as efficient and simple as possible. Do not use phase 1 or big M method, you should start from the feasible solution given!

Consider the set of arcs E for which $x_{ij} > 0$. We wish to construct a feasible tree solution from E. There can be two problems: The first problem is that the graph of all nodes and the edges E may not be connected. This is easy to fix, add to E as few arcs as possible with 0 flow to get connectivity.

The second problem is that the arcs of E may contain cycles. This can be fixed similar to the step in the Network Simplex Method, in which a new arc is added to the tree, creating a cycle. To get rid of cycles in E, we first find any cycle C in E. Pick any arc (i,j) in the cycle, and decrease the flow of arcs in C in the direction of (i,j) by t, and increasing the flow of arcs in the cycle in the opposite direction by t (similar to what is done in the network simplex method). Pick $t =$ smallest $x_{k,l}$ in the cycle to be decreased. After this change, remove from E the arc of the cycle whose flow became 0. (If more than one arc of the cycle now has flow 0, we remove only one such arc.) Now C is no longer a cycle in the new E. Repeat this process, until all cycles in E dissapear. Since the initial number of arcs in E is at most n^2, and the final number of arcs should be $n - 1$, this step is repeated at most $n^2 - n + 1$ times.

5). (15 points) Consider the following knapsack problem:

$$\begin{aligned} \max \quad & z = x_1 + x_2 + 8x_3 + 2x_4 + 5x_5 \\ & 3x_1 + 3x_2 + 4x_3 + 4x_4 + 2x_5 \leq 7 \\ & 0 \leq x_1, x_2, x_3, x_4, x_5 \leq 1 \end{aligned}$$

(a). Set up the starting simplex tableau with x_2, x_3, x_4 non basic at the lower bound and x_1, x_5 non basic at the upper bound, s the slack variable is basic.

z	x_1	x_2	x_3	x_4	x_5	s	rhs
1	-1	-1	-8	-2	-5	0	6
0	3	3	4	4	2	1	2

(b). Using the Simplex method with upper bounds, starting from the tableau in part (a), which variable enters the basis, what will be the value of this new basic variable, and which leaves the basis?

At this point any of the three variables at their lower bound could enter the basis. I picked x_3 since is objective coefficient is as negative as possible (Dantzig's rule for a max problem.) Using the min ratio test we get $\Delta_3 = \min\{2/4, \infty, 1\} = 2/4$. $s = 0$ and leaves the basis. The new $z = 10$. The new tableau is:

z	x_1	x_2	x_3	x_4	x_5	s	rhs
1	5	5	0	6	-1	2	10
0	3/4	3/4	1	1	1/2	1/4	1/2

This new tableau is not yet optimal. Next x_1 will enter the basis. x_3 will become non basic at its upper bound, etc.

(c). Now consider the general (fractional) knapsack problem:

$$\begin{aligned} \max \quad & z = c_1x_1 + \ldots + c_nx_n \\ & a_1x_1 + \ldots + a_nx_n \leq b \\ & 0 \leq x_1, \ldots, x_n \leq 1 \end{aligned}$$

where $c_i > 0$, $a_i > 0$ for $i = 1, \ldots, n$ and $b > 0$. Show that for all such knapsack problems, there exists an optimal solution in which at most one of the variables is non integer. Hint: Consider BFS.

A BFS contains exactly one variable, since there is one constraint. All other variables are non basic, and therefore equal to their lower or upper bounds, 0 or 1, both integer. Thus only the basic variable may be non integer.

6). (15 points) We wish to design a simpler version of the Ellipsoid Algorithm for solving problems of the sort: Does there exist a vector x for which $Ax < b$? (As in class, we assume data is integral.) Recall, we know from class that if there exists a feasible x, then there also exists an x feasible to $Ax < b$ and $(x-0)^t(x-0) < n4^L$ (a sphere centered at the origin). Suppose we also know that if the feasible region $F = \{x \mid Ax < b, (x-0)^t(x-0) < n4^L\}$ is non empty, then it contains a sphere with volume at least V for some $V > 0$. Below I describe a simple *sphere algorithm*. I use the notation A^i is the ith row of matrix A.

Sphere Algorithm:

1. k=0, $S_0 = \{x \mid (x-0)^t(x-0) < n4^L\}$, $t_0 = 0$
2. If $At_k < b$ stop and report t_k is feasible.
3. otherwise, let i be a violated constraint, $A^i t_k \geq b_i$.
4. define $\frac{1}{2}S_k = S_k \cap \{x \mid A^i x \leq A^i t_k\}$
5. find the smallest sphere $S_{k+1} \supseteq \frac{1}{2}S_k$
6. if volume of of $S_{k+1} < V$ stop and report the problem is infeasible.
7. $k = k + 1$, goto step 2.

(a). Each of the steps 1-7 can easily be implemented to run in polynomial time. Unfortunately, the Sphere algorithm does not solve the problem in polynomial time. Explain why. Hint: Think about the key lemma.

The key lemma states that the volume of the new ellipsoid is at most the volume of the previous ellipsoid times some constant $f < 1$. This implies that after k steps, the volume of the current ellipsoid is at most f^k times the volume of the initial ellipsoid, which we can bound, and so we can bound the number of steps before an ellipsoid must have volume that is "too small" by some polynomial.

However, for the algorithm described here, the only way to get a new sphere containing $\frac{1}{2}S_k$ is for the new sphere $S_{k+1} = S_k$, and so our spheres are not shrinking at all, and the algorithm makes no progress (it is an infinite loop!).

(b). We try to fix the algorithm by replacing $\frac{1}{2}S_k$ in step 4, with $S'_k = S_k \cap \{x \mid A^i x \leq b_i\}$. Note that $S'_k \subset \frac{1}{2}S_k$. In step 5, we find the smallest sphere S_{k+1} containing S'_k (which is easy to do in polynomial time). Does this modified algorithm solve the problem in polynomial time? Explain.

No. Since S'_k can be arbitrarily close to $\frac{1}{2}S_k$, even though the new sphere S_{k+1} will have volume less than or equal to the previous sphere, S_k, we cannot prove a lemma equivalent to the key lemma which says $vol(S_{k+1}) \leq f \cdot vol(S_k)$ for some $f < 1$, thus we cannot get a polynomial bound on the number of steps until the volume of a sphere becomes smaller than V.