AMS 540 / MBA 540 (Fall, 2008) Estie Arkin

# Linear Programming - Final

Do all problems. Write your answers on the exam. You are permitted to use the text, your notes and any material handed out in class. The exam time is 2 hours and 30 minutes. GOOD LUCK!!!

Academic integrity is expected of all students at all times, whether in the presence or absence of members of the faculty. Understanding this, I declare that I shall not give, use, or receive unauthorized aid in this examination.

**Name (PRINT CLEARLY), ID number and Signature:**

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**1).** (10 points) Consider the LP

$$\begin{aligned} \max \quad z &= 4x_1 + x_2 \\ 3x_1 + 2x_2 &\leq 6 \\ 6x_1 + 3x_2 &\leq 10 \\ x_1, x_2 &\geq 0 \end{aligned}$$

I solved the problem and got that the objective row of the optimal tableau is

| $z$ | $x_1$ | $x_2$ | $s_1$ | $s_2$ | RHS |
|---|---|---|---|---|---|
| 1 | 0 | 2 | 0 | 1 | 20/3 |

where $s_1$, $s_2$ are the slack variables in the first and second constraints respectively. Prove using duality, that my computation is wrong! Note: Do *not* solve the problem "from scratch", and show that you got a different answer. Use duality!

**2).** (30 points) The following LP describes a company that makes 2 products, with 3 constraints. The optimal tableau is given below. (The big M method was used to solve the problem, $s_3$ is the slack of the third constraint, $e_2$ is the excess variable of the second constraint, and $a_1$, $a_2$ are the artificial variables of the first and second constraints.) Answer each part using sensitivity analysis. Do *not* solve from scratch! Show your work. Each part is independent of the others.

$$\begin{aligned} \max \quad & z = 4x_1 + x_2 \\ & x_1 + 2x_2 = 6 \\ & x_1 - x_2 \geq 3 \\ & 2x_1 + x_2 \leq 10 \\ & x_1, x_2 \geq 0 \end{aligned}$$

| $z$ | $x_1$ | $x_2$ | $e_2$ | $s_3$ | $a_1$ | $a_2$ | RHS |
|---|---|---|---|---|---|---|---|
| 1 | 0 | 0 | 0 | 7/3 | M-2/3 | M | 58/3 |
| 0 | 0 | 1 | 0 | -1/3 | 2/3 | 0 | 2/3 |
| 0 | 1 | 0 | 0 | 2/3 | -1/3 | 0 | 14/3 |
| 0 | 0 | 0 | 1 | 1 | -1 | -1 | 1 |

(a). What is the dual of this LP?

(b). What is the most that the company should be willing to pay for an extra unit of resource 3?

(c). Find the range of values for $b_3$ (the RHS of the third constraint, which is 10 now) for which the current basis remains optimal.

(d). A new constraint is added: $x_1 + x_2 \leq 5$. Clearly the old optimal solution is not feasible to this new constraint. Using sensitivity analysis and dual Simplex, add this new constraint to the tableau and find the new optimal tableau. (Do not solve from scratch!)

**3).** (20 points) Consider the LP $\max\{cx \mid Ax \le b,\ x \ge 0\}$. Let $x^*$ be an optimal soltuion. Suppose that $A$ is decomposed into $\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$ and $b$ is decomposed into $\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ such that $A_1x^* = b_1$ and $A_2x^* < b_2$. Show that $x^*$ is also an optimal solution to the problem LP$_1$: $\max\{cx \mid A_1x \le b_1,\ x \ge 0\}$.

**4).** (10 points) Consider a transshipment problem: $\min\{\sum_i \sum_j c_{ij}x_{ij} \mid \sum_j x_{ji} - \sum_j x_{ij} = b_i,\ x \ge 0\}$. Suppose we use the following method to find a starting solution: If all supplies and demands are equal to zero, stop. A feasible solution is given with no flow on the edges. Otherwise, pick any node $i$ with a supply $s_i > 0$ $(b_i = -s_i)$, and any node $j$ with demand $b_j > 0$, find a directed path in the network from $i$ to $j$, and ship on it the minimum of $s_i, b_j$ (meaning add this flow to the amount currently being sent on each edge of this path). Decrease the supply and demand on $i$ and $j$, noting that one (or both) of these became zero. Repeat the procedure until all supplies and demands become zero.
Prove or give a counter example: The feasible solution found by this method must be a Feasible Tree Solution (equivalently, a BFS).

**5).** (15 points) A small town draws its water supply from the local (small) river. The flow of water in the river fluctuates with the seasons. To meet higher demand for water in some seasons, the town must store water in its reservoir during periods of high runoff. There is a storage cost of $ 10 per quarter (season) per acre feet of water. (Water used in the same quarter that it was drawn from the river has no storage costs.) In addition, the water has to be treated for human consumption. The treatment costs, supply and demand are given in the table below. (Assume the water must be treated in the quarter it is used.)

| Season | Demand (acre-feet) | Supply (acre-feet) | Treatment cost ($) |
|---|---|---|---|
| Jan-Mar | 150 | 300 | 25 |
| Apr-Jun | 200 | 900 | 40 |
| Jul-Sep | 600 | 220 | 30 |
| Oct-Dec | 400 | 60 | 35 |

Formulate the problem of meeting the demand for water at minimum cost as a Balanced Transportation Problem. Give your formulation using a cost and requirement table.

**6).** (15 points) We have seen in class that a transshipment problem in which the sum of supplies exceeds the sum of demands can easily be transform into a problem in which the sums are equal (by adding a dummy node whose demand is equal to the surplus with shipping costs to dummy equal to zero). Let $z^*$ be the optimal cost to this transshipment problem. Assume $c_{ij} \geq 0$ for all $i, j$.

(a). Suppose that we increase the supply of one node by 1 unit, (and modify the demand at the dummy node to re establish supply equal demand) holding all other data fixed, and let $z(a)$ be the new optimal solution. Show that $z^* \geq z(a)$.

(b). Suppose that instead we increase the demand of one node by 1 unit, (and modify the demand at the dummy node to re establish supply equal demand) holding all other data fixed, and let $z(b)$ be the new optimal solution. Show that $z^* \leq z(b)$.