

**Linear Programming - Final**

Do all problems. Write your answers on the exam. You are permitted to use the text, your notes and any material handed out in class. The exam time is 2 hours and 30 minutes. GOOD LUCK!!!

Academic integrity is expected of all students at all times, whether in the presence or absence of members of the faculty. Understanding this, I declare that I shall not give, use, or receive unauthorized aid in this examination. I have been warned that if I am caught cheating (either receiving or giving unauthorized aid) I will get the grade "Q" for this course.

**Name (PRINT CLEARLY), ID number and Signature:**

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**1).** (10 points) Consider a transshipment problem defined in class, in which  $b_i$  is the demand/supply of node  $i$ ,  $\sum_{i=1}^n b_i = 0$ , and all  $c_{ij}$  and  $b_i$  are integral.

$$\begin{aligned} \min \quad & z = \sum_{i=1}^n \sum_{j=1}^n c_{ij} x_{ij} \\ & \sum_{j=1}^n x_{ji} - \sum_{k=1}^n x_{ik} = b_i, \quad i = 1, \dots, n \\ & x_{ij} \geq 0, \quad i, j = 1, \dots, n \end{aligned}$$

Prove or give a counterexample:

(a). If all supplies and demands are even ( $b_i$  is even for all  $i$ ), then for some optimal solution  $x^*$  all  $x_{ij}^*$  are even.

(b). If all supplies and demands are odd ( $b_i$  is odd for all  $i$ ), then for some optimal solution  $x^*$  all  $x_{ij}^*$  are odd.

2). (40 points) A Company manufactures 3 types of candy bars, each made up entirely of chocolate and sugar. 50 oz of sugar and 100 oz of chocolate are available. The compositions of each candy bar and profit are given below.

	oz of sugar	oz of chocolate	profit (cents)
Bar 1	1	2	3
Bar 2	1	3	7
Bar 3	1	1	5

Define  $x_i$  to be the number of candy bar type  $i$  made, we get the following LP:

$$\begin{aligned}
 \max \quad & z = 3x_1 + 7x_2 + 5x_3 \\
 & x_1 + x_2 + x_3 \leq 50 \\
 & 2x_1 + 3x_2 + x_3 \leq 100 \\
 & x_1, x_2, x_3 \geq 0
 \end{aligned}$$

Let the slack variables  $s_1$  and  $s_2$ . The optimal tableau was obtained:

	$z$	$x_1$	$x_2$	$x_3$	$s_1$	$s_2$	RHS
$z$	1	3	0	0	4	1	300
$x_3$	0	$\frac{1}{2}$	0	1	$\frac{3}{2}$	$-\frac{1}{2}$	25
$x_2$	0	$\frac{1}{2}$	1	0	$-\frac{1}{2}$	$\frac{1}{2}$	25

(a). Find the dual of this LP and its optimal solution (the objective value and the value of the dual variables). Use the tableau - do not solve from scratch!

(b). What is the most that the company should be willing to pay for an extra ounce of sugar?

(c). Find the range of values of the objective function coefficient for  $x_1$  for which the current basis remains optimal.

(d). Find the range of values of the RHS coefficient  $b_1$  (amount of sugar) for which the current basis remains optimal.

(e). Suppose the amount of sugar available was changed to 80, and the amount of chocolate available was changed to 140, is the current BFS still optimal? Explain.

(f). Suppose candy bar 1 requires only 0.5 oz sugar and 0.5 oz chocolate. Find the new optimal solution, using sensitivity analysis (do not solve from scratch!).

**3).** (15 points) Consider the LP

$$\begin{aligned} \min \quad & z = cx - b^t w \\ & Ax \geq b \\ & -A^t w \geq -c^t \\ & x, w \geq 0 \end{aligned}$$

Where  $A$  is an  $m$  by  $n$  matrix,  $b$  is an  $m$  by 1 matrix (i.e., a column vector with  $m$  components), and  $c$  is a 1 by  $n$  matrix (i.e., a row vector with  $n$  components).  $x$  and  $w$  are vectors of variables. Prove that if the LP is feasible, then its has an optimum objective function equal to zero ( $z = 0$ ).

4). (15 points) There are three school districts in a city. The Supreme Court requires the schools to be racially balanced. District 1 has 210 nonminority students and 120 minority students. District 2 has 210 nonminority students and 30 minority students. District 3 has 180 nonminority students and 150 minority students. Each school district must have exactly 300 students, and all must have the same number of nonminority students, and same number of minority students. The distance between district 1 and 2 is 3 miles, the distance between district 1 and 3 is 5 miles and the distance between district 2 and 3 is 4 miles. Formulate a Balanced Transportation Problem to minimize the total distance traveled by students while satisfying the Supreme Court's ruling. Give your formulation in terms of a cost and requirement table. Recall that a Transportation Problem is *Balanced* if the sum of the supplies is equal to the sum of the demands.

5). (10 points) Consider an upper bounded transshipment problem defined in class, in which  $b_i$  is the demand/supply of node  $i$ ,  $\sum_{i=1}^n b_i = 0$ , and all  $c_{ij}$  and  $b_i$  are integral.

$$\begin{aligned} \min \quad & z = \sum_{i=1}^n \sum_{j=1}^n c_{ij} x_{ij} \\ & \sum_{j=1}^n x_{ji} - \sum_{k=1}^n x_{ik} = b_i, \quad i = 1, \dots, n \\ & x_{ij} \geq 0, \quad i, j = 1, \dots, n \end{aligned}$$

Assume that the problem has a unique optimal solution  $x^*$ . Prove or give a counterexample: (One obvious counterexample is letting all the  $b_i$  equal to zero, so to make the problem non boring, you should assume that some of the  $b_i$ 's are *not* equal to zero.)

(a). Let  $(k, l)$  be an arc of minimum cost in the graph. The optimal solution must have  $x_{kl}^* > 0$ ?

(b). Let  $(p, q)$  be an arc of maximum cost in the graph. The optimal solution must have  $x_{pq}^* = 0$ ?

6). (10 points) The following LP is being solved using the simplex method for bounded variables:

$$\begin{aligned} \max \quad & z = 4x_1 + 3x_2 + 5x_3 \\ \text{s.t.} \quad & 2x_1 + 2x_2 + x_3 + x_4 \leq 9 \\ & 4x_1 - x_2 - x_3 + x_5 \leq 6 \\ & 2x_2 + x_3 \leq 6 \\ & 0 \leq x_1 \leq 2 \\ & 0 \leq x_2 \leq 3 \\ & 0 \leq x_3 \leq 4 \\ & 0 \leq x_4 \leq 5 \\ & 0 \leq x_5 \leq 7 \end{aligned}$$

Add slack variables  $s_1$ ,  $s_2$  and  $s_3$  to constraints 1,2 and 3. Let  $s_1, s_2, s_3$  be the basic variables,  $x_1, x_3$  non basic at their upper bounds and  $x_2, x_4, x_5$  non basic at their lower bounds. What is the corresponding tableau?