AMS 540 / MBA 540 (Fall, 2007) Estie Arkin

Linear Programming - Midterm solution sketch

Mean 66.76, median 68, high 100 (2 of them), low 25.

1). (18 points) The following is the first tableau of phase I for an LP, the artificial variables a_1, a_2, a_3 were added to constraints 1,2,3 respectively. Let $a_1 = x_7$, $a_2 = x_8$, $a_3 = x_9$.

w	x_1	x_2	x_3	x_4	x_5	x_6	a_1	a_2	a_3	RHS
1	0	0	1	0	2	-1	0	0	0	3
0	1	-1	1	0	2	0	1	0	0	0
0	-2	1	0	0	-2	0	0	1	0	3
0	1	0	1	0	1	-1	0	0	1	0
0	0	2	1	1	2	1	0	0	0	4

(a). Using Bland's rule, which variable enters the basis, and which variable leaves the basis? x_3 enters, (smaller index than x_5) and $a_1 = x_7$ leaves (ties with $a_3 = x_9$ for min ratio test, but has smaller index).

Common mistakes: saying that phase is a max problem, it is not! Saying a_1 or a_3 leaves, No! In Bland's rule and lex rule, there are no "or"s, there is a unique choice for leaving variable!

Note: the current basic variables (which can be identified as having a col of the identity matrix) are a_1, a_2, a_3, x_4, so onlt those can leave the basis! A variable not in the basis (such as x_1 or x_2) cannot leave it at this step!

(b). Using the lexicographic rule, which variable enters the basis, and which variable leaves the basis? Either x_3 or x_5 enters. If we use max $z_j - c_j$ we choose x_5. We form the lex-min-ratio vectors, using $B^{-1} = B = I$ which can be found under a_1, a_2, a_3, x_4: For a_1 $(1/2)(0, 1, 0, 0, 0)$ for a_2 no ratio since $y_{25} \leq 0$, for a_3, $(1/1)(0, 0, 0, 1, 0)$ and for x_4, $(1/2)(4, 0, 0, 0, 1)$, so the winner is a_3.

Common mistake: Forgetting b_i in the lex ratio test, doing a lex-max test (it should always be lex-min), forgetting a col of B^{-1}.

(c) Explain why phase I will usually have multiple (alternative) optimal solutions. Every feasible solution to the original LP is optimal for the phase I LP, and so it will have (usually) multiple optimal solutions.

Common mistakes: using degeneracy, or the fact that the current (non optimal) tableau has some $z_j - c_j = 0$ for non basic variables.

2). (25 points) During the next 2 months General Cars must meet (on time) the following demands for trucks and cars; month 1 - 300 trucks, 600 cars; month 2 - 300 trucks, 300 cars. During each month at most 1000 vehicles can be produced. Each truck uses 2 tons of steel, and each car uses 1 ton of steel. During month 1, steel costs \$400 per ton; during month 2 steel costs \$600 per ton. At most 1500 tons of steel may be purchased each month. Steel may only be used during the month it is purchased. At the end of each month a holding cost of \$150 per vehicle is assessed. Each car gets 20 mpg, and each truck gets 10 mpg. During each month, the vehicles produced by the company must average at least 16 mpg. Formulate an LP to meet the demand at minimum cost. Make sure to clearly define your variables!

Let T_1, T_2 be the number of trucks produced in months 1,2. Similarly C_1, C_2. Let IC, IT be the inventory of cars and trucks from month 1 to month 2.

$$\begin{aligned}
\min \quad & z = 400(2T_1 + C_1) + 600(2T_2 + C_2) + 150(IT + IC) \\
\text{s. t.} \quad & T_1 + C_1 \leq 1000 \\
& T_2 + C_2 \leq 1000 \\
& 2T_1 + C_1 \leq 1500 \\
& 2T_2 + C_2 \leq 1500 \\
& 20C_1 + 10T_1 \geq 16(C_1 + T_1) \\
& 20C_2 + 10T_2 \geq 16(C_2 + T_2) \\
& T_1 = 300 + IT \\
& C_1 = 600 + IC
\end{aligned}$$

$$IT + T_2 = 300$$
$$IC + C_2 = 300$$
$$C_1, C_2, T_1, T_2, IC, IT \geq 0$$

Common mistake: using constraints $C_1 \geq 600$, $T_1 \geq 300$, $C_2 \geq 300$ and $T_2 \geq 300$. The last 2 constraints assume that there will be no inventroty from month 1 to month 2, so all the demand must be met by production in that month. (The first 2 constraints are ok.) Another mistake - defining s_1 = steel used in month 1, $s_1 \leq 1500$ but no constraint connecting the amount of steel used to make vehicles with steel bought, so we can make vehicles, but buy zero steel.

3). (30 points) Consider an LP: (p_1 and p_2 are given constants.)

$$\begin{aligned} \max \quad z = -p_1 x_1 + p_2 x_2 \\ x_1 - x_2 = 0 \\ 0 \leq x_1, x_2 \leq 1 \end{aligned}$$

(a). What are the extreme points and extreme directions of this LP? Extreme points are $v_1 = (0,0)$ and $v_2 = (1,1)$, there are no extreme directions (the region is a line segment, which is bounded!).

(b). Write the equivalent LP in terms of the extreme points and extreme directions. $c = (-p_1, p_2)$, $cv_1 = 0$, $cv_2 = -p_1 + p_2$

$$\begin{aligned} \max \quad z = \lambda \cdot 0 + (1-\lambda)(-p_1 + p_2) \\ 0 \leq \lambda \leq 1 \end{aligned}$$

(c). What is the dual of the (original) LP? Rewrite the original problem as:

$$\begin{aligned} \max \quad z = -p_1 x_1 + p_2 x_2 \\ x_1 - x_2 = 0 \\ x_1 \leq 1 \\ x_2 \leq 1 \\ x_1, x_2 \geq 0 \end{aligned}$$

So the dual has 3 variables and 2 constraints:

$$\begin{aligned} \min \quad w = 0w_1 + w_2 + w_3 \\ w_1 + w_2 \geq -p_1 \\ -w_1 + w_3 \geq p_2 \\ w_1 \text{ unr }, w_2, w_3 \geq 0 \end{aligned}$$

Common mistake: thinking the primal is

$$\begin{aligned} \max \quad z = -p_1 x_1 + p_2 x_2 \\ x_1 - x_2 = 0 \\ x_2 \leq 1 \\ x_1 \geq 0 \end{aligned}$$

(d). Now suppose that $p_1 > p_2$. What is the optimal solution to the LP? (Make sure to state z as well as x_1, x_2.) In this case $-p_1 + p_2 < 0$ so we see from part (b) that the optimal solution is $\lambda = 1$, $z = 0$, $x_1 = x_2 = 0$.

(e). What is the optimal solution to the dual LP (that you gave in part (c))? Make sure to state the optimal objective function as well as the value of the dual variables. By the corollary to weak duality, if we show a dual (feasible) solution with $w = z = 0$, we know it is optimal to the dual problem. So we can let $w_2 = w_3 = 0$, and get $-p_1 \leq w_1 \leq -p_2$.

4). (15 points) Consider the LP (P) $\min\{cx \mid Ax = b,\ x \geq 0\}$. (As usual, assume that A is an m by n matrix and b is a vector of dim m.) Let B be a basis for an optimal soltuion. Now consider a new LP, (P(λ)) $\min\{cx \mid Ax = b + \lambda d,\ x \geq 0\}$, where λ is a scalar and d is a fixed non zero vector of dimension m, in other words, replace b by $b + \lambda d$.

(a). Prove that if B is an optimal (and feasible) basis for the original problem (i.e., when $\lambda = 0$) and also optimal (and feasible) for some $\lambda' > 0$ then B is an optimal (and feasible) basis for all $0 \leq \lambda \leq \lambda'$. Note that λ only appears in the right hand side. So, we know that $B^{-1}b \geq 0$ and $B^{-1}(b + \lambda' d) \geq 0$, we want to show that $B^{-1}(b + \lambda d) \geq 0$ for all $0 \leq \lambda \leq \lambda'$. So we define $\alpha = \lambda/\lambda'$, and know that $0 \leq \alpha \leq 1$ by definition.

$(1-\alpha)B^{-1}b \geq 0$ and $(\alpha)B^{-1}(b+\lambda' d) \geq 0$, we add these up and get $(1-\alpha)B^{-1}b + (\alpha)B^{-1}b + \alpha\lambda' B^{-1}d \geq 0$ implying that $B^{-1}b + \lambda B^{-1}d \geq$ and we are done.

(b). Give a condition such that the basis B will be optimal (and feasible) for all $\lambda \geq 0$. We need to check feasibility, $x_b = B^{-1}(b + \lambda d) \geq 0$ for all λ, which implies that $B^{-1}d \geq 0$.

5). (12 points) Consider an LP $\min\{cx \mid Ax = b,\ e^t x = 1,\ x \geq 0\}$, where $e \in R^n$ is a vector of ones. Suppose we are given a tableau of a BFS for the LP, x', which objective value z_0. Since I am lazy, I do not wish to continue pivoting, but I would like to know how far my current solution z_0 can be from the optimal solution z^*. Give a lower bound on z^* in terms of the data in the current tableau. (In other words, your lower bound may only use the terms $\bar{b}$, y_{ik}, $z_j - c_j$, z_0, R.)

Define $F = \max(0, \max_{j\in R}(z_j - c_j))$. Recall that for any feasible solution, we know that $z = z_0 - \sum_{j\in R}(z_j - c_j)x_j$. Using the fact that $z_j - c_j \leq F$ by our definition, which implies that $-(z_j - c_j) \geq -F$, and that $\sum_j x_j = 1$, we get that

$$
\begin{aligned}
z &= z_0 - \sum_{j\in R}(z_j - c_j)x_j \\
&\geq z_0 - \sum_{j\in R} F \cdot x_j \\
&= z_0 - F\sum_{j\in R} x_j \\
&\geq z_0 - F
\end{aligned}
$$

Note that $F \geq 0$ and that if $F = 0$ then all the $z_j - c_j \leq 0$ at this tableau, and $z^* = z_0$. (See midterm 2005 for a very similar problem.)