AMS/MBA 546 (Spring, 2007) Estie Arkin

# Network Flows - Final

Do 4 of the following 6 problems. You are permitted to use the text (AMO), your notes and any material handed out in class. The exam time is 2 hours and 30 minutes. GOOD LUCK!!!

Academic integrity is expected of all students at all times, whether in the presence or absence of members of the faculty. Understanding this, I declare that I shall not give, use, or receive unauthorized aid in this examination. I have been warned that if I am caught cheating (either receiving or giving unauthorized aid) I will get the grade "Q" for this course.

**Name (PRINT CLEARLY), and Signature:**

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**Problems to be graded:**

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**1).** Prove: A graph $G = (V, E)$ has a vertex cover which is also an independent set if and only if $G$ is a bipartite graph. (Definitions: $C \subset V$ is a vertex cover if for every edge $(i, j) \in E$ either $i \in C$ or $j \in C$ or both. $I \subset V$ is an independent set if for every edge $(i, j) \in E$ at most one of the nodes $i, j$ is in $I$.)

**2).** Consider a *min cost flow problem* on $G = (N, A)$ with capacities $u_{ij}$ supply/demand $b(i)$ and costs $c_{ij}$:

$$\begin{array}{lllll} \min & z = \sum_{(i,j)\in A} c_{ij}x_{ij} & & & \\ \text{s.t.} & \sum_j x_{ij} - \sum_j x_{ji} & = b(i) & \text{for all } i \in N \\ & 0 \le x_{ij} & \le u_{ij} & \text{for all } (i,j) \in A \end{array}$$

(a). Let $(k, l)$ be an arc of minimum cost in the graph. Is it possible that no min cost flow for this has $x_{kl} > 0$? (One obvious such example is a graph for which all the $b(i)$ are equal to zero, so to make the problem non boring, you should assume that some of the $b(i)$ are *not* equal to zero.)

(b). Let $(p, q)$ be an arc of maximum cost in the graph.Is it possible that in every min cost flow on this graph $x_{pq} > 0$?

(c). Suppose some $u_{ij} = \infty$ and some $c_{ij} < 0$. Show that the optimal solution $z$ is finite ($z > -\infty$) if and only if the arcs with capacities $c_{ij} = \infty$ do not contain a negative cost cycle.

**3).** In set $S$ of $nk$ ($k \neq 0$) balls each ball has one of $n$ distinct colours and has one of $n$ distinct diameters. Furthermore, each colour is represented by exactly $k$ balls and each diameter is represented by exactly $k$ balls.

(a). We wish to select the largest set of balls such that every diameter and colour is represented at most once. Show how to formulate the problem as a maximum matching problem.

(b). Show that in fact $n$ such balls can be selected (i.e., $n$ balls such that every diameter and colour is represented exactly once). Hint: Use Hall's Theorem which states: In a bipartite graph $G = (N_1, N_2, E)$

where $|N_1| = |N_2|$, the graph has a perfect matching if and only if for every subset $X \subseteq N_1$ we have $|X| \leq |\Gamma(X)|$, where $\Gamma(X)$ are the neighbours of $X$.

**4).** The Clustering TSP problem is: Given a complete undirected graph $G = (V, E)$ with non negative costs on the edges satisfying the triangle inequality. The nodes are partitioned into "clusters" $V = V_1 \cup V_2 \cup ... V_k$, $V_i \cap V_j = \emptyset$ for $i \neq j$. The salesman wants to visit all nodes at minimum total cost, with the additional restriction that nodes of a cluster be traversed consecutively.

(a). Prove that the Clustering TSP problem is NP-hard.

(b). Describe an approximation algorithm for the clustering TSP. You should clearly describe an algorithm that runs in polynomial time, then prove that the tour you obtain has length at most some constant times the length of the optimal clustering TSP tour, for all instances of the problem.

**5).** An *orientation* of an undirected graph $G = (V, E)$ is a directed graph $D = (V, A)$, such that $A$ contains exactly one of $(u, v)$ and $(v, u)$ for every edge $\{u, v\} \in E$. Prove that $\chi(G) \leq p + 1$ if and only if there exists an orientation $D$ of $G$ in which the longest directed path has length at most $p$. Hints: For one direction, assume a colouring with $\chi(G)$ colours exists and show how to direct the edges. For the second direction, given a direction of the edges, temporarily remove arcs so that the resulting graph is acyclic, show how to colour the nodes, and that this colouring works also once the removed arcs are put back into the graph.

**6).** Given a directed graph $G = (N, A)$ with non negative lengths on the arcs $c_{ij}$ and two specified nodes $s, t \in N$. A *vital arc* is an arc whose removal from the graph causes the length of the shortest path from $s$ to $t$ to increase. A *most vital arc* is a vital arc whose removal results in the largest increase in the shortest path length from $s$ to $t$. For parts (a),(b),(c) prove or give a counterexample. Part (d) asks for an algorithm.

(a). A most vital arc is an arc with maximum value of $c_{ij}$ on some shortest path from $s$ to $t$.

(b). An arc that does not belong to any shortest path from $s$ to $t$ cannot be a most vital arc.

(c). A graph might contain several most vital arcs.

(d). Describe an algorithm for determining a most vital arc in a graph, or showing that none exists. What is the running time of your algorithm?