AMS/MBA 546 (Spring, 2008) Estie Arkin

Network Flows - Final

Do 4 of the following 6 problems. You are permitted to use the text (AMO), your notes and any material handed out in class. The exam time is 2 hours and 30 minutes. GOOD LUCK!!!

Academic integrity is expected of all students at all times, whether in the presence or absence of members of the faculty. Understanding this, I declare that I shall not give, use, or receive unauthorized aid in this examination.

Name (PRINT CLEARLY), and Signature:

Problems to be graded:

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1). Consider a set of n numbers $a_1, a_2, \ldots, a_n$ arranged in non-decreasing order of their values ($a_i \leq a_{i+1}$). We wish to partition these numbers into "clusters" so that (1) each cluster contains at least p numbers; (2) each cluster contains consecutive numbers from the list $a_1, a_2, \ldots, a_n$; and (3) the sum of the squared deviation of the numbers from their cluster mean is as small as possible.

Let $\bar{a}(S) = (\sum_{i \in S} a_i)/|S|$ denote the mean of a set S of numbers forming a cluster. For a number a_k in the cluster, its squared deviation from the cluster mean is $(a_k - \bar{a}(S))^2$.

Describe how to formulate this problem as a shortest path problem. Make sure to clearly define the nodes and arcs of your graph, the costs on the arcs, and what are s and t.

2). (a). Given a tree on $n > 2$ nodes, with exactly 2 nodes of degree equal to 1, show that all other nodes of this tree must have degree equal to 2.

(b). Let G be a graph on n nodes, such that both G and its complement are trees. What is n. (Make sure to justify your answer!)

3). We are given a directed graph $G = (N, A)$ with arc costs c_{ij}, and we know that no negative cost cycles exist. Define f_{ij} to be the maximum amount we can decrease the cost of arc (i, j) without creating any negative cycles, assuming that all other arc costs remain the same. Describe an efficient algorithm for determining f_{ij} for all arcs $(i, j) \in A$.

4). We are given a directed graph $G = (N, E)$, a source node s and sink node t with (nonnegative) capacities on the arcs u_{ij}. We consider the MAX FLOW problem on this graph. A *most vital arc* is defined as an arc whose deletion causes the largest decrease in the maximum flow value v. A *least vital arc* is defined as an arc whose deletion causes the smallest decrease in the maximum flow value v.

Prove or give a counterexample:

(a). A most vital arc is an arc with the maximum value of u_{ij}.

(b). A most vital arc is an arc with the maximum value of x_{ij}.

(c). Any arc with the $x_{ij} = 0$ in any maximum flow is a least vital arc.

(d). Any arc that does belongs to some minimum cut cannot be a least vital arc.

5). We are given a connected graph $G = (V, E)$, with n nodes and nonnegative costs on the edges. Define a 1-TREE to be a connected subset of n edges $E' \subset E$ that spans V. (In other words, a 1-TREE is a spanning tree of G plus one more edge.)

(a). Describe an efficient (polynomial time) algorithm to find a MINIMUM COST 1-TREE. Don't forget to **prove** that the output of your algorithm is indeed a MINIMUM COST 1-TREE! You may use any of the algorithms described in class, and you do not have to reprove their correctness.

(b). Prove that the cost of a MINIMUM COST 1-TREE is less than or equal to the cost of any TRAVELING SALESMAN TOUR on G.

6). The RURAL POSTMAN PROBLEM is: Given a graph $G = (V, E)$ with non negative costs on the edges, and a subset of the edges $E' \subset E$. The postman must to find a closed walk including all edges of E' at minimum total cost.

Describe an approximation algorithm for the RURAL POSTMAN. You should clearly describe an algorithm that runs in polynomial time, then prove that the closed walk you obtain has length at most some constant times the length of the optimal closed walk, for all instances of the problem.