Structure of optimal policies to periodic-review inventory models with convex costs and backorders for all values of discount factors

Eugene A. Feinberg · Yan Liang

Abstract This paper describes the structure of optimal policies for discounted periodic-review single-commodity total-cost inventory control problems with fixed ordering costs for finite and infinite horizons. There are known conditions in the literature for optimality of \((s_t, S_t)\) policies for finite-horizon problems and the optimality of \((s, S)\) policies for infinite-horizon problems. The results of this paper cover the situation, when such assumption may not hold. This paper describes a parameter, which, together with the value of the discount factor and the horizon length, defines the structure of an optimal policy. For the infinite horizon, depending on the values of this parameter and the discount factor, an optimal policy either is an \((s, S)\) policy or never orders inventory. For a finite horizon, depending on the values of this parameter, the discount factor, and the horizon length, there are three possible structures of an optimal policy: (1) it is an \((s_t, S_t)\) policy, (2) it is an \((s_t, S_t)\) policy at earlier stages and then does not order inventory, or (3) it never orders inventory. The paper also establishes continuity of optimal value functions and describes alternative optimal actions at states \(s_t\) and \(s\).

Keywords Inventory control · Finite horizon · Infinite horizon · Optimal policy · \((s, S)\) policy

1 Introduction

It is well-known that, for the classic periodic-review single-commodity inventory control problems with fixed ordering costs, \((s, S)\) policies are optimal for the expected total cost criterion under certain conditions on cost functions. These policies order up to the level \(S\),
when the inventory level is less than \( s \), and do not order otherwise. This paper investigates the general situations, when \((s, S)\) policies may not be optimal.

Systematic studies of inventory control problems started with the papers by Arrow et al. (1951) and Dvoretzky et al. (1953). Most of the earlier results are surveyed in the books by Bensoussan (2011), Beyer et al. (2010), Porteus (2002), Simchi-Levi et al. (2004), and Zipkin (2000), see also Katehakis et al. (2016) and Shi et al. (2013) for recent results for continuous review models.

Recently developed general optimality conditions for discrete-time Markov Decision Processes (MDPs) applicable to inventory control problem are discussed in the tutorial by Feinberg (2016). Here, we mention just a few directly relevant references. Scarf (1960) introduced the concept of \( K \)-convexity to prove the optimality of \((s, S)\) policies for finite-horizon problems with continuous demand. Zabel (1962) indicated some gaps in Scarf (1960) and corrected them. Iglehart (1963) extended the results in Scarf (1960) to infinite-horizon problems with continuous demand. Veinott and Wagner (1965) proved the optimality of \((s, S)\) policies for both finite-horizon and infinite-horizon problems with discrete demand. Zheng (1991) provided an alternative proof for discrete demand. Beyer and Sethi (1999) completed the missing proofs in Iglehart (1963) and Veinott and Wagner (1965). In general, \((s, S)\) policies may not be optimal. To ensure the optimality of \((s, S)\) policies, the additional assumption on backordering cost function (see Condition 3.3 below) is used in many papers including Iglehart (1963) and Veinott and Wagner (1965). Relevant assumptions are used in Schäl (1976), Heyman and Sobel (1984), Bertsekas (2000), Chen and Simchi-Levi (2004a, b), Huh and Janakiraman (2008), and Huh et al. (2011). As shown by Veinott and Wagner (1965) for problems with discrete demand and Feinberg and Lewis (2015) for an arbitrary distributed demand, such assumptions are not needed for an infinite-horizon problem, when the discount factor is close to 1.

For problems with linear holding costs, according to Simchi-Levi et al. (2004, Theorem 8.3.4, p. 126), finite-horizon undiscounted value functions are continuous and, according to Bensoussan (2011, Theorem 9.11, p. 118), infinite-horizon discounted value functions are continuous. These continuity properties obviously hold, if the amounts of stored inventory are limited only to integer values, and they are nontrivial if the amounts of stored inventory are modeled by real-valued numbers. General results on MDPs state only lower semi-continuity of discounted value functions; see Feinberg et al. (2012, Theorem 2).

This paper studies the structure of optimal policies without the assumption on backordering costs mentioned above. We describe a parameter, which, together with the value of the discount factor and the horizon length, defines the structure of an optimal policy. For a finite horizon, depending on the values of this parameter, the discount factor, and the horizon length, there are three possible structures of an optimal policy: (1) it is an \((s_t, S_t)\) policy, (2) it is an \((s_t, S_t)\) policy at earlier stages and then does not order inventory, or (3) it never orders inventory. For the infinite horizon, depending on the values of this parameter and the discount factor, an optimal policy either is an \((s, S)\) policy or never orders inventory. This paper also establishes continuity of optimal discounted value functions for finite and infinite-horizon problems. The continuity of values functions is used to prove that, if the amount of stored inventory is modeled by real numbers, then ordering up to the levels \(S_t\) and \(S\) are also optimal actions at states \(s_t\) and \(S_t\) respectively for discounted finite and infinite-horizon problems; see Corollary 5.4 below.

The rest of the paper is organized in the following way. Section 2 introduces the classic stochastic periodic-review single-commodity inventory control problems with fixed ordering costs. Section 3 presents the known results on the optimality of \((s, S)\) policies. Section 4 describes the structure of optimal policies for finite-horizon and infinite-horizon problems.
2 Model definition

Let $\mathbb{R}$ denote the real line, $\mathbb{Z}$ denote the set of all integers, $\mathbb{R}_+ := [0, +\infty)$ and $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$. Consider the classic stochastic periodic-review inventory control problem with fixed ordering cost and general demand. At times $t = 0, 1, \ldots$, a decision-maker views the current inventory of a single commodity and makes an ordering decision. Assuming zero lead times, the products are immediately available to meet demand. Demand is then realized, the decision-maker views the remaining inventory, and the process continues. The unmet demand is backlogged and the cost of inventory held or backlogged (negative inventory) is modeled as a convex function. The demand and the order quantity are assumed to be non-negative.

The state and action spaces are either (1) $X = \mathbb{R}$ and $A = \mathbb{R}_+$, or (2) $X = \mathbb{Z}$ and $A = \mathbb{N}_0$.

The inventory control problem is defined by the following parameters.

1. $K > 0$ is a fixed ordering cost;
2. $\bar{c} > 0$ is the per unit ordering cost;
3. $h(\cdot)$ is the holding/backordering cost per period, which is assumed to be a convex real-valued function on $X$ such that $h(x) \to \infty$ as $|x| \to \infty$; without loss of generality, consider $h(\cdot)$ to be non-negative;
4. $\{D_t, t = 1, 2, \ldots\}$ is a sequence of i.i.d. non-negative finite random variables representing the demand at periods $0, 1, \ldots$. We assume that $\mathbb{E}[D] < +\infty$ and $\mathbb{P}(D > 0) > 0$, where $D$ is a random variable with the same distribution as $D_1$;
5. $\alpha \geq 0$ is the discount factor for finite-horizon problems, and $\alpha \in [0, 1)$ for infinite-horizon problems.

Without loss of generality, assume that $h(0) = 0$. The assumption $\mathbb{P}(D > 0) > 0$ avoids the trivial case when there is no demand. Define $S_0 := 0$ and

$$S_t := \sum_{j=1}^t D_j, \quad t = 1, 2, \ldots. \quad (2.1)$$

Then $\mathbb{E}[S_t] = t \mathbb{E}[D] < +\infty$ for all $t = 0, 1, \ldots$.

Define the following function for all $y, z \in X$ such that $y \neq z$,

$$H(y, z) := \frac{h(y) - h(z)}{y - z}. \quad (2.2)$$

The convexity of $h$ implies that $H(y, z)$ is non-decreasing in $y$ on $X\setminus\{z\}$ for all $z \in X$ and non-decreasing in $z$ on $X\setminus\{y\}$ for all $y \in X$; see Hiriart-Urruty and Lemaréchal (1993, Proposition 1.1.4 on p. 4).

Since $\frac{h(x)}{x} = H(x, 0)$ is a non-decreasing function on $X\setminus\{0\}$, then consider the limit

$$k_h := -\lim_{x \to -\infty} \frac{h(x)}{x}. \quad (2.3)$$

Since $h(x) \to \infty$ as $x \to -\infty$, then there exists $x^* < 0$ such that $h(x^*) > 0$. Therefore, $H(x^*, 0) < 0$. Thus $0 < k_h \leq +\infty$.

The dynamics of the system is defined by the equations

$$x_{t+1} = x_t + a_t - D_{t+1}, \quad t = 0, 1, 2, \ldots,$$
where \( x_t \) and \( a_t \) denote the current inventory level and the ordered amount at period \( t \) respectively. If an action \( a \) is chosen at state \( x \) then the following cost is collected,

\[
c(x, a) = K I_{\{a > 0\}} + \tilde{c}a + \mathbb{E}[h(x + a - D)], \quad (x, a) \in \mathbb{X} \times \mathbb{A},
\]

(2.4)

where \( I_{\{a > 0\}} \) is an indicator of the event \( \{a > 0\} \).

Let \( H_t = (\mathbb{X} \times \mathbb{A})^t \times \mathbb{X} \) be the sets of histories up to periods \( t = 0, 1, \ldots \). A (randomized) decision rule at period \( t = 0, 1, \ldots \) is a regular transition probability \( \pi_t : H_t \rightarrow \mathbb{A} \); that is, (1) \( \pi_t(\cdot| h_t) \) is a probability distribution on \( \mathbb{A} \), where \( h_t = (x_0, a_0, x_1, \ldots, a_{t-1}, x_t) \), and (2) for any measurable subset \( B \subseteq \mathbb{A} \), the function \( \pi_t(B|\cdot) \) is measurable on \( H_t \). A policy \( \pi \) is a sequence \( (\pi_0, \pi_1, \ldots) \) of decision rules. Moreover, \( \pi \) is called non-randomized if each probability measure \( \pi_t(\cdot| h_t) \) is concentrated at one point. A non-randomized policy is called Markov if all decisions depend only on the current state and time. A Markov policy is called stationary if all decisions depend only on the current state.

For a finite horizon \( N = 0, 1, \ldots \) and a discount factor \( \alpha \geq 0 \), define the expected total discounted costs

\[
v_{N, \alpha}^\pi(x) := \mathbb{E}_x^\pi \left[ \sum_{t=0}^{N-1} \alpha^t c(x_t, a_t) \right].
\]

(2.5)

When \( N = +\infty \) and \( \alpha \in [0, 1) \), (2.5) defines the infinite horizon expected total discounted cost of \( \pi \) denoted by \( v_{\infty, \alpha}^\pi(x) \) instead of \( v_{\infty, \alpha}^\pi(x) \). Define the optimal values

\[
v_{N, \alpha}(x) = \inf_{\pi \in \Pi} v_{N, \alpha}^\pi(x), \quad \text{and} \quad v_{\alpha}(x) = \inf_{\pi \in \Pi} v_{\infty, \alpha}^\pi(x),
\]

where \( \Pi \) is the set of all policies. A policy \( \pi \) is called optimal for the respective criterion if \( v_{N, \alpha}^\pi(x) = v_{N, \alpha}(x) \) or \( v_{\alpha}^\pi(x) = v_{\alpha}(x) \) for all \( x \in \mathbb{X} \).

3 Optimality of \((s, S)\) policies

It is known that optimal \((s, S)\) policies may not exists. This section considers the known sufficient condition for the optimality of \((s_t, \alpha, s_t, \alpha)\) and \((s_\alpha, S_\alpha)\) policies for discounted problems.

The value functions for the inventory control problem defined in Sect. 2 can be written as

\[
v_{t+1, \alpha}(x) = \min_{a \geq 0} \left\{ KL_{\{a > 0\}} + G_{t, \alpha}(x + a) \right\} - \tilde{c}x, \quad t = 0, 1, \ldots,
\]

(3.1)

\[
v_{\alpha}(x) = \min_{a \geq 0} \left\{ KL_{\{a > 0\}} + G_{\alpha}(x + a) \right\} - \tilde{c}x,
\]

(3.2)

and

\[
G_{t, \alpha}(x) = \tilde{c}x + \mathbb{E}[h(x - D)] + \alpha \mathbb{E}[v_{t, \alpha}(x - D)], \quad t = 0, 1, \ldots,
\]

(3.3)

\[
G_{\alpha}(x) = \tilde{c}x + \mathbb{E}[h(x - D)] + \alpha \mathbb{E}[v_{\alpha}(x - D)].
\]

(3.4)

and \( v_{0, \alpha}(x) = 0 \) for all \( x \in \mathbb{X}, \alpha \geq 0 \) in equalities (3.1), (3.3), and \( \alpha \in [0, 1) \) in equalities (3.2), (3.4); e.g. see Feinberg and Lewis (2015). The functions \( G_{t, \alpha} \) and \( G_{\alpha} \) are lower semicontinuous for all \( t = 0, 1, \ldots \) and \( \alpha \in [0, 1) \); Feinberg and Lewis (2015, Corollary 6.4). Since all the costs are nonnegative, equalities (3.3) and (3.4) imply that

\[
\lim_{x \to +\infty} G_{t, \alpha}(x) = \lim_{x \to +\infty} G_{\alpha}(x) = +\infty, \quad t = 0, 1, \ldots.
\]

(3.5)

Recall the definitions of \( K \)-convex functions and \((s, S)\) policies.
Definition 3.1 A function $f : \mathbb{X} \rightarrow \mathbb{R}$ is called $K$-convex, where $K \geq 0$, if for each $x \leq y$ and for each $\lambda \in (0, 1)$,

$$f((1 - \lambda)x + \lambda y) \leq (1 - \lambda)f(x) + \lambda f(y) + \lambda K.$$ 

Suppose $f$ is a lower semi-continuous $K$-convex function, such that $f(x) \rightarrow \infty$ as $|x| \rightarrow \infty$. Let

$$S \in \arg\min_{x \in \mathbb{X}} \{f(x)\},$$

$$s = \inf\{x \leq S : f(x) \leq K + f(S)\}. \quad (3.6)$$

Definition 3.2 Let $s_t$ and $S_t$ be real numbers such that $s_t \leq S_t$, $t = 0, 1, \ldots$. A policy is called an $(s_t, S_t)$ policy at step $t$ if it orders up to the level $S_t$, if $x_t < s_t$, and does not order, if $x_t \geq s_t$. A Markov policy is called an $(s_t, S_t)$ policy if it is an $(s_t, S_t)$ policy at all steps $t = 0, 1, \ldots$. A policy is called an $(s, S)$ policy if it is stationary and it is an $(s, S)$ policy at all steps $t = 0, 1, \ldots$.

Condition 3.3 (cp. Veinott and Wagner 1965) There exist $z, y \in \mathbb{X}$ such that $z < y$ and

$$\frac{h(y) - h(z)}{y - z} < -\bar{c}. \quad (3.8)$$

It is well-known that for the problem considered in this paper and for relevant problems, this condition and its variations imply optimality $(s_t, S_t)$ policies for finite-horizon problems and $(s, S)$ policies for infinite-horizon problems; see Scarf (1960), Iglehart (1963) and Veinott and Wagner (1965). The following theorem presents the result from Feinberg and Lewis (2015) for finite and infinite horizons and for arbitrary demand distributions; see also Chen and Simchi-Levi (2004a, b), if price is fixed for the coordinating inventory control and pricing problems considered there.

Theorem 3.4 (Feinberg and Lewis 2015, Theorem 6.12) If Condition 3.3 is satisfied, then the following statements hold:

(i) Let $\alpha \geq 0$. For $t = 0, 1, \ldots$ consider real numbers $S_{t, \alpha}$ satisfying (3.6) and $s_{t, \alpha}$ defined in (3.7) with $g(x) = G_{t, \alpha}(x)$, $x \in \mathbb{X}$. Then for every $N = 1, 2, \ldots$ the $(s_{t}, S_{t})$ policy with $s_{t} = s_{N-t-1, \alpha}$ and $S_{t} = S_{N-t-1, \alpha}$, $t = 0, 1, \ldots, N - 1$, is optimal for the $N$-horizon problem.

(ii) Let $\alpha \in [0, 1)$. Consider real numbers $S_{\alpha}$ satisfying (3.6) and $s_{\alpha}$ defined in (3.7) for $g(x) := G_{\alpha}(x)$, $x \in \mathbb{R}$. Then the $(s_{\alpha}, S_{\alpha})$ policy is optimal for the infinite-horizon problem with the discount factor $\alpha$. Furthermore, a sequence of pairs $\{(s_{t, \alpha}, S_{t, \alpha})\}_{t=0,1,\ldots}$ considered in statement (i) is bounded, and, if $(s^{*}_{\alpha}, S^{*}_{\alpha})$ is a limit point of this sequence, then the $(s^{*}_{\alpha}, S^{*}_{\alpha})$ policy is optimal for the infinite-horizon problem.

If Condition 3.3 does not hold, then finite-horizon optimal $(s_{t, \alpha}, S_{t, \alpha})$ policies may not exist. It is shown Veinott and Wagner (1965) for discrete demand distributions and by Feinberg and Lewis (2015) for arbitrary demand distributions that finite-horizon optimal $(s_{t, \alpha}, S_{t, \alpha})$ policies exist if certain non-zero terminal costs are assumed.

Theorem 3.5 (Feinberg and Lewis 2015, Theorem 6.10) There exists $\alpha^{\ast} \in [0, 1)$ such that an $(s_{\alpha}, S_{\alpha})$ policy is optimal for the infinite-horizon expected total discounted cost criterion with a discount factor $\alpha \in (\alpha^{\ast}, 1)$, where the real numbers $S_{\alpha}$ satisfy (3.6) and $s_{\alpha}$ are defined
by (3.7) with \( f(x) = G_\alpha(x), x \in X \). Furthermore, a sequence of pairs \((s_t, \alpha), s_t, \alpha)_{t=0,1,...}\), where the real numbers \(s_t, \alpha\) satisfy (3.6) and \(s_t, \alpha\) are defined in (3.7) with \( f(x) = G_t, \alpha(x), x \in X \), is bounded, and, for each its limit point \((s^*_\alpha, S^*_\alpha)\), the \((s^*_\alpha, S^*_\alpha)\) policy is optimal for the infinite-horizon problem with the discount factor \(\alpha\).

4 Structure of optimal polices

This section describes the structure of finite-horizon and infinite-horizon optimal policies. Unlike the previous section, it covers the situations when \((s, S)\) policies are not optimal. Define

\[
\alpha^* := 1 - \frac{k_h}{\bar{c}},
\]

where \(k_h\) is introduced in (2.3). Since \(0 < k_h \leq +\infty\), then \(-\infty \leq \alpha^* < 1\).

**Lemma 4.1** Condition 3.3 holds if and only if \(\alpha^* < 0\), which is equivalent to \(k_h > \bar{c}\).

**Proof** The inequalities \(\alpha^* < 0\) and \(k_h > \bar{c}\) are equivalent because of \(\bar{c} > 0\). Since \(k_h > 0\), it is sufficient to prove that Condition 3.3 does not hold if and only if \(k_h \in (0, \bar{c}]\). Consider the function \(H(y, z)\) defined in (2.2) for \(y, z \in X\) such that \(y \neq z\).

Let Condition 3.3 do not hold. Then we have \(H(y, z) \geq -\bar{c}\) for all \(z < y\). Since \(H(0, x)\) is non-decreasing and bounded below by \(-\bar{c}\) when \(x < 0\), then \(-\bar{c} \leq \lim_{x \to -\infty} H(0, x)\). Therefore, \(k_h = -\lim_{x \to -\infty} H(0, x) \in (0, \bar{c}]\) in view of (2.3).

Now, let us prove that \(k_h \in (0, \bar{c}]\) implies that Condition 3.3 does not hold. Formula (2.3) implies that \(\lim_{x \to -\infty} H(y, z) = -k_h\) for all \(y \in X\). Since \(H(y, z)\) is non-decreasing in \(z\) for \(z < y\), then \(H(y, z) \geq -k_h \geq -\bar{c}\) for all \(y, z \in X\) satisfying \(z < y\). Therefore, Condition 3.3 does not hold. \(\square\)

Define the following function for all \(t \in \mathbb{N}_0\) and \(\alpha \geq 0\),

\[
f_{t, \alpha}(x) := \bar{c}x + \sum_{i=0}^{t} \alpha^i \mathbb{E}[h(x - S_{i+1})], \quad x \in X.
\]

Observe that \(f_{0, \alpha}(x) = \bar{c}x + \mathbb{E}[h(x - D)] = G_{0, \alpha}\). Since \(h(x)\) is a convex function, then the function \(f_{t, \alpha}(x)\) is convex for all \(t \in \mathbb{N}_0\) and \(\alpha \geq 0\).

Let \(f_{t, \alpha}(-\infty) := \lim_{x \to -\infty} f_{t, \alpha}(x)\) and

\[
N_\alpha := \inf\{t \in \mathbb{N}_0 : f_{t, \alpha}(-\infty) = +\infty\},
\]

where the infimum of an empty set is \(+\infty\). Since the function \(h(x)\) is non-negative, then the function \(f_{t, \alpha}(x)\) is non-decreasing in \(t\) for all \(x \in X\) and \(\alpha \geq 0\). Therefore, (1) \(N_\alpha\) is non-increasing in \(\alpha\), that is, \(N_\alpha \leq N_\beta\), if \(\alpha > \beta\); and (2) in view of the definition of \(N_\alpha\), for each \(t \in \mathbb{N}_0\)

\[
f_{t, \alpha}(-\infty) < +\infty, \quad \text{if } t < N_\alpha, \quad \text{and } f_{t, \alpha}(-\infty) = +\infty, \quad \text{if } t \geq N_\alpha.
\]

The following theorem provides the complete description of optimal finite-horizon policies for all discount factors \(\alpha\).
Table 1 The structure of optimal policies for a discounted $N$-horizon problem with $N < +\infty$ and $\alpha \geq 0$

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\alpha^* &lt; 0$</th>
<th>$0 \leq \alpha^* &lt; \alpha$</th>
<th>$1 &gt; \alpha^* \geq \alpha$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>For the natural number $N_{\alpha}$ defined in (4.3), the policy $(s_t, S_t)$ policy at steps $t = N - N_{\alpha}, \ldots, N - 1$, and is an $(s_t, S_t)$ policy prior to that, is optimal.</td>
<td>The policy that never orders is optimal.</td>
<td>The policy that never orders is optimal.</td>
</tr>
<tr>
<td></td>
<td>if $N &gt; N_{\alpha}$, then a policy, that never orders at steps $t = N - N_{\alpha}, \ldots, N - 1$ and is an $(s_t, S_t)$ policy at steps $t = 0, \ldots, N - N_{\alpha} - 1$, is optimal; if $N \leq N_{\alpha}$, then a policy that never orders is optimal.</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Fig. 1 The structure of optimal policies for a discounted $N$-horizon problem with $N < +\infty$ and $\alpha \in [0, 1)$

**Theorem 4.2** Let $\alpha > 0$. Consider $\alpha^*$ defined in (4.1). If $\alpha^* < 0$ (that is, Condition 3.3 is satisfied), then the statement of Theorem 3.4(i) holds. If $0 \leq \alpha^* < 1$, then the following statements hold for the finite-horizon problem with the discount factor $\alpha$:

(i) if $\alpha \in [0, \alpha^*]$, then the policy that never orders is optimal for every finite horizon $N = 1, 2, \ldots$;

(ii) if $\alpha > \alpha^*$, then $N_{\alpha} < +\infty$ and, for a finite horizon $N = 1, 2, \ldots$, the following statements hold:

(a) if $N \leq N_{\alpha}$, then the policy that never orders is optimal;

(b) if $N > N_{\alpha}$, then a policy, that never orders at steps $t = N - N_{\alpha}, N - N_{\alpha} + 1, \ldots, N - 1$, and is an $(s_t, S_t)$ policy with $s_t = s_{N - t - 1, \alpha}$ and $S_t = S_{N - t - 1, \alpha}$ at steps $t = 0, 1, \ldots, N - N_{\alpha} - 1$, is optimal, where the real numbers $S_{N - t - 1, \alpha}$ satisfy (3.6) and $s_{N - t - 1, \alpha}$ are defined in (3.7) with $f(x) := G_{N - t - 1, \alpha}(x), x \in X$.

**Remark 4.3** For the $N$-horizon inventory control problem, according to Simchi-Levi et al. (2004, Theorem 8.3.4, p. 126), $(s_t, S_t)$ policies, $t = 0, 1, \ldots, N$, are optimal. However, in the model considered there, the inventory left at time $N$ has a salvage value $\bar{c}$ per unit. Furthermore, in that formulation, $\alpha = 1$ implies that the value of $\bar{c}$ does not affect the decisions. Let us take $\bar{c} > 0$ small enough to have $\alpha^* < 0$. Then theorem 4.2 also implies the optimality of $(s_t, S_t)$ policies, $t = 0, 1, \ldots, N$.

The conclusions of Theorem 4.2 are presented in Table 1. If the discount factor $\alpha \in [0, 1)$, the conclusions of Theorem 4.2 are presented in Fig. 1, in addition, if the discount factor $\alpha \geq 1$, then the case presented in the last column of the Table 1, that is if $1 > \alpha^* \geq \alpha$, is impossible, and the conclusions for $\alpha \geq 1$ are presented in Fig. 2.

The following theorem describes optimal infinite-horizon policies for all discount factors $\alpha \in [0, 1)$.

**Theorem 4.4** Let $\alpha \in [0, 1)$. Consider $\alpha^*$ defined in (4.1). The following statements hold for the infinite-horizon problem with the discount factor $\alpha$:
There is an optimal \((s_t, S_t)\) policy.

For \(N > N_\alpha\) a policy, that does not order on the last \(N_\alpha\) steps and is an \((s_t, S_t)\) policy prior to that, is optimal.

For \(N \leq N_\alpha\) the policy that never orders is optimal.

Fig. 2 The structure of optimal policies for a discounted \(N\)-horizon problem with \(N < +\infty\) and \(\alpha \geq 1\)

<table>
<thead>
<tr>
<th>(\alpha)</th>
<th>(\alpha^* &lt; \alpha)</th>
<th>(\alpha \leq \alpha^*)</th>
</tr>
</thead>
<tbody>
<tr>
<td>There is an optimal ((s, S)) policy</td>
<td>The policy that never orders is optimal</td>
<td></td>
</tr>
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</table>

Fig. 3 The structure of optimal policies for a discounted infinite-horizon problem with \(\alpha \in [0, 1)\)

(i) \(\alpha^* < \alpha\), then an \((s_\alpha, S_\alpha)\) policy is optimal, where the real numbers \(S_\alpha\) and \(s_\alpha\) are defined in (3.6) and (3.7) respectively with \(f(x) := G_\alpha(x), x \in \mathbb{X}\). Furthermore, a sequence of pairs \((s_t, S_t, S_\alpha)\) considered in Theorem 4.2 (ii,b) is bounded, and, for if \((s_\alpha^*, S_\alpha^*)\) is a limit point of the sequence, then the \((s_\alpha^*, S_\alpha^*)\) policy is optimal.

(ii) \(\alpha^* \geq \alpha\), then the policy that never orders is optimal.

The conclusions of Theorem 4.4 are presented in Table 2 and Fig. 3. To prove Theorems 4.2 and 4.4, we first establish several auxiliary statements.

**Lemma 4.5** If Condition 3.3 does not hold, then:

(i) \(\mathbb{E}[h(x - S_t)] \leq \tilde{c}t\mathbb{E}[D] - \tilde{c}x\) for all \(x \leq 0\) and \(t = 0, 1, \ldots\);

(ii) for each \(\epsilon \in (0, k_h)\) there exists a number \(M_\epsilon < 0\) such that for all \(z < y \leq M_\epsilon\) and \(t = 0, 1, \ldots\),

\[
-k_h \leq \frac{\mathbb{E}[h(y - S_t) - h(z - S_t)]}{y - z} < -k_h + \epsilon < 0.
\]

**Proof** (i) Consider the function \(H(y, z)\) defined in (2.2) for all \(y, z \in \mathbb{X}\) satisfying \(y \neq z\). According to Lemma 4.1, since Condition 3.3 does not hold, then \(k_h \in (0, \tilde{c}]\). Since \(H(y, z)\) is non-decreasing in \(y\) on \(\mathbb{X}\) for all \(z \in \mathbb{X}\) and non-decreasing in \(z\) on \(\mathbb{X}\) for all \(y \in \mathbb{X}\), then \(H(0, x) = \frac{h(x)}{x} \geq \tilde{c}\) for all \(x < 0\), which is equivalent to \(h(x) \leq \tilde{c}x\) for all \(x \leq 0\). Let \(x \leq 0\). Then \(x - S_t \leq 0\) almost surely (a.s.) for all \(t = 0, 1, \ldots\). Thus \(\mathbb{E}[h(x - S_t)] \leq \mathbb{E}[-\tilde{c}(x - S_t)] = \tilde{c}\mathbb{E}[D] - \tilde{c}x\) for all \(x \leq 0\) and \(t = 0, 1, \ldots\).

(ii) Since \(\lim_{x \to -\infty} H(0, x) = -k_h\) and \(H(0, x)\) is non-decreasing when \(x < 0\), then for each \(\epsilon \in (0, k_h)\) there exists \(M_\epsilon < 0\) such that \(-k_h \leq H(0, M_\epsilon) < -k_h + \epsilon < 0\). Therefore, \(H(y, z) \leq H(0, z) \leq H(0, M_\epsilon) < -k_h + \epsilon\) for all \(z < y \leq M_\epsilon\), where the first two inequalities follow from the monotonicity properties of \(H(y, z)\) stated in the first
paragraph of the proof. As follows from (2.3), lim_{z \to -\infty} H(y, z) = -k_h. Since the function $H(y, z)$ is non-decreasing in $z$ when $z < y$, for all $z < y \leq M_\epsilon$,

$$-k_h \leq H(y, z) = \frac{h(y) - h(z)}{y - z} < -k_h + \epsilon.$$  \hfill (4.6)

Since $S_t \geq 0$ a.s. for all $t = 0, 1, \ldots$, then (4.6) implies that $-k_h \leq H(y - S_t, z - S_t) < -k_h + \epsilon$ a.s. for all $z < y \leq M_\epsilon$, which yields $-k_h \leq \mathbb{E}[H(y - S_t, z - S_t)] < -k_h + \epsilon$. The last inequalities are equivalent to (4.5).

**Lemma 4.6** If the function $G_{t, \alpha}(x)$ is convex in $x$ and $\lim_{x \to -\infty} G_{t, \alpha}(x) < +\infty$ for some $t = 0, 1, \ldots$, then for the epoch $t$ the minimum in the optimality equation (3.1) is achieved for all $x \in \mathbb{R}$ at the action $a = 0$, and the functions $v_{t+1,\alpha}(x)$ and $G_{t+1,\alpha}(x)$ are convex in $x$.

**Proof** Since $G_{t, \alpha}(x)$ is a convex function and $\lim_{x \to -\infty} G_{t, \alpha}(x) < +\infty$, then the function $G_{t, \alpha}(x)$ is non-decreasing on $\mathbb{R}$. Therefore, $G_{t, \alpha}(x) \leq K + G_{t, \alpha}(x + a)$ for all $x \in \mathbb{R}$ and $a \geq 0$. In view of (3.1), the action $a = 0$ is optimal at the epoch $t$. Therefore, $v_{t+1,\alpha}(x) = G_{t, \alpha}(x) - \tilde{c}x$. This formula and convexity of the function $G_{t, \alpha}(x)$ imply that $v_{t+1,\alpha}(x)$ is a convex function, which, in view of (3.3), implies that the function $G_{t+1,\alpha}(x)$ is convex. \hfill $\square$

**Lemma 4.7** Let $\alpha > 0$ and there exists $t_0 = 0, 1, \ldots$ such that the function $G_{t_0,\alpha}(x)$ is $K$-convex and $\lim_{x \to -\infty} G_{t_0,\alpha}(x) = +\infty$. Then the functions $G_{t,\alpha}(x)$, $t = t_0, t_0 + 1, \ldots$, are $K$-convex and $G_{t,\alpha}(x) \to +\infty$ as $|x| \to +\infty$.

**Proof** Since all the costs are nonnegative, $v_{t,\alpha}(x) \leq v_{t+1,\alpha}(x)$ and therefore $G_{t,\alpha}(x) \leq G_{t+1,\alpha}(x)$ for all $x \in \mathbb{R}$ and for all $t = 0, 1, \ldots$. This implies $\lim_{|x| \to +\infty} G_{t,\alpha}(x) = +\infty$ for $t = t_0, t_0 + 1, \ldots$. Assume that the function $G_{t,\alpha}(x)$ is $K$-convex for some $t \geq t_0$. In view of Heyman and Sobel (1984, Lemma 7-2), since $\lim_{|x| \to +\infty} G_{t,\alpha}(x) = +\infty$ and $G_{t,\alpha}(x)$ is $K$-convex, then the function $v_{t+1,\alpha}(x)$ is $K$-convex. Therefore, (3.3) implies that the function $G_{t+1,\alpha}(x)$ is $K$-convex. Since $G_{t_0,\alpha}(x)$ is $K$-convex, then the induction arguments imply that the functions $G_{t,\alpha}(x)$ are $K$-convex for all $t = t_0, t_0 + 1, \ldots$. \hfill $\square$

**Lemma 4.8** Let $\alpha \in (0, 1)$ and there exists $t_0 = 0, 1, \ldots$ such that the function $G_{t_0,\alpha}(x)$ is $K$-convex and $\lim_{x \to -\infty} G_{t_0,\alpha}(x) = +\infty$. Then the function $G_{\alpha}(x)$ is $K$-convex and $G_{\alpha}(x) \to +\infty$ as $|x| \to +\infty$.

**Proof** In view of Lemma 4.7, the functions $G_{t,\alpha}(x)$, $t = t_0, t_0 + 1, \ldots$, are $K$-convex and $G_{t,\alpha}(x) \to +\infty$ as $|x| \to +\infty$. According to Feinberg et al. (2012, Theorem 2), $v_{t,\alpha}(x) \uparrow v_{\alpha}(x)$ as $t \to +\infty$ and therefore $G_{t,\alpha}(x) \uparrow G_{\alpha}(x)$ as $t \to +\infty$. The $K$-convexity of the functions $G_{t,\alpha}$ stated in Lemma 4.7 implies the $K$-convexity of the function $G_{\alpha}$. \hfill $\square$

**Lemma 4.9** For $\alpha \geq 0$ consider the function $f_{t,\alpha}(x)$ and number $N_{\alpha}$ defined in (4.2) and (4.3). The following statements hold:

(i) if $N_{\alpha} < +\infty$, then $G_{t,\alpha}(x) = f_{t,\alpha}(x)$ for all $t = 0, 1, \ldots, N_{\alpha}$;

(ii) if $N_{\alpha} = +\infty$, then $G_{t,\alpha}(x) = f_{t,\alpha}(x)$ for all $t = 0, 1, \ldots$.

**Proof** Let us prove this lemma by induction. As stated after formula (4.2), $G_{0,\alpha}(x) = f_{0,\alpha}(x), x \in \mathbb{R}$. Now assume that $G_{k,\alpha}(x) = f_{k,\alpha}(x), x \in \mathbb{R}$, for some $k \in \mathbb{N}_0$ satisfying $k < N_{\alpha}$. Then
\[ G_{k+1, \alpha}(x) = \tilde{c}x + \mathbb{E}[h(x - D)] + \alpha \mathbb{E}[G_{k, \alpha}(x - D) - \tilde{c}(x - D)] \]
\[ = \tilde{c}x + \mathbb{E}[h(x - S_1)] + \alpha \mathbb{E}[f_{k, \alpha}(x - D) - \tilde{c}(x - D)] \]
\[ = \tilde{c}x + \sum_{i=0}^{k+1} \alpha^i \mathbb{E}[h(x - S_{i+1})] = f_{k+1, \alpha}(x), \]  
(4.7)

where the first equality follows from Lemma 4.6 and Eqs. (3.3), (4.4), the second one follows from (4.5), the first equality and the last inequality are straightforward, and the last equality follows from the definition of \( \alpha^* \). Hence the induction arguments imply the conclusions in statements (i) and (ii). \( \square \)

**Lemma 4.10** Consider \( \alpha^* = 1 - \frac{kh}{c} \) defined in (4.1). Let Condition 3.3 do not hold. Then the following statements hold:

(i) If \( \alpha > \alpha^* \), then \( 1 \leq N_\alpha < +\infty \);
(ii) if \( \alpha \in [0, \alpha^*] \), then \( N_\alpha = +\infty \) and, in addition, the function \( G_\alpha(x) \) is convex and \( \lim_{x \to -\infty} G_\alpha(x) < +\infty \).

**Proof** According to Lemma 4.1, since Condition 3.3 does not hold, then \( \alpha^* \geq 0 \).

(i) If \( \alpha > \alpha^* \), then there exists \( \delta > 0 \) such that \( \alpha > \alpha^* + \delta \) and \( \epsilon \delta < k_h \). Let \( \epsilon_k = \tilde{c}\delta \in (0, k_h) \). Then \( 0 \leq \alpha^* \leq \alpha + \delta = 1 + \frac{\epsilon_k - k_h}{c} < \alpha \). According to Lemma 4.5(ii), for \( \epsilon_k \in (0, k_h) \), there exists \( M_{\epsilon_k} < 0 \) such that (4.5) holds for all \( z < y \leq M_{\epsilon_k} \) and \( t = 0, 1, \ldots \). Therefore, for all \( z < y \leq M_{\epsilon_k} \) and \( t = 0, 1, \ldots \),
\[ \sum_{i=0}^{t} \alpha^i \mathbb{E}[h(y - S_{i+1}) - h(z - S_{i+1})] \leq (-k_h + \epsilon_k) \sum_{i=0}^{t} \alpha^i < 0. \]  
(4.8)

If \( \alpha \in (\alpha^*, 1) \), then \( (-k_h + \epsilon_k) \sum_{i=0}^{+\infty} \alpha^i = - \frac{k_h + \epsilon_k}{1 - \alpha} < - \frac{k_h + \epsilon_k}{1 - (\alpha^* + \delta)} = -\tilde{c} \). If \( \alpha \geq 1 \), then \( (-k_h + \epsilon_k) \sum_{i=0}^{+\infty} \alpha^i = - \infty < -\tilde{c} \). Therefore, for all \( \alpha > \alpha^* \) there exists a natural number \( M \) such that
\[ (-k_h + \epsilon_k) \sum_{i=0}^{M} \alpha^i < -\tilde{c}. \]  
(4.9)

Thus, (4.8) and (4.9) imply that there exist \( y, z \in \mathbb{X} \) satisfying \( z < y \) such that
\[ \sum_{i=0}^{M} \alpha^i \mathbb{E}[h(y - S_{i+1}) - h(z - S_{i+1})] < -\tilde{c}, \]  
which is equivalent to
\[ f_{M, \alpha}(y) - f_{M, \alpha}(z) < 0. \]  
(4.10)

Since the function \( f_{M, \alpha}(x) \) is convex, then (4.10) implies that \( f_{M, \alpha}(\infty) = +\infty \). Therefore, \( N_\alpha \leq M < +\infty \). Since Condition 3.3 does not hold, then \( f_{0, \alpha}(\infty) = +\infty \). Therefore, \( N_\alpha \geq 1 \).

(ii) Consider \( \alpha \in [0, \alpha^*] \). According to Lemma 4.5(ii), for \( \epsilon > 0 \), there exists \( M_\epsilon < 0 \) such that for all \( z < y \leq M_\epsilon \) and \( t = 0, 1, \ldots \),
\[ \sum_{i=0}^{t} \alpha^i \mathbb{E}[h(y - S_{i+1}) - h(z - S_{i+1})] \geq \sum_{i=0}^{+\infty} \alpha^i \mathbb{E}[h(y - S_{i+1}) - h(z - S_{i+1})] \geq -k_h \sum_{i=0}^{+\infty} \alpha^i = - \frac{k_h}{1 - \alpha} \geq - \frac{k_h}{1 - \alpha^*} = -\tilde{c}, \]  
(4.11)

where the first two inequalities follow from (4.5), the first equality and the last inequality are straightforward, and the last equality follows from the definition of \( \alpha^* \). In view of (4.2), (4.11)
equivalent to \( f_{t, \alpha}(y) \geq f_{t, \alpha}(z) \) for all \( t \in \mathbb{N}_0 \) and \( z < y \leq M_\epsilon \). Therefore, \( f_{t, \alpha}(-\infty) < +\infty \) for all \( t = 0, 1, \ldots \), which implies that \( N_\alpha = +\infty \).

According to Feinberg et al. (2012, Theorem 2), \( v_{t, \alpha}(x) \uparrow v_\alpha(x) \) as \( t \to +\infty \) and therefore \( G_{t, \alpha}(x) \uparrow G_\alpha(x) \) as \( t \to +\infty \). Therefore, in view of Lemma \( 4.9(ii) \),

\[
G_\alpha(x) = \tilde{c}x + \sum_{i=0}^{+\infty} \alpha^i \mathbb{E}[h(x - S_{i+1})],
\]

which implies that the function \( G_\alpha(x) \) is convex. Observe that \( G_\alpha(0) \leq \sum_{i=0}^{+\infty} \alpha^i \tilde{c}(i + 1)\mathbb{E}[D] = \tilde{c}\mathbb{E}[D] \frac{\alpha(1-\alpha)}{(1-\alpha^2)} < +\infty \), where the first inequality follows from (4.12) and Lemma 4.5(i). In view of (4.11), it is equivalent to \( G_\alpha(y) \geq G_\alpha(z) \) for all \( z < y \leq M_\epsilon \). Therefore, since the function \( G_\alpha(x) \) is convex, \( \lim_{x \to -\infty} G_\alpha(x) < +\infty \).

\[
\square
\]

\textbf{Proof of Theorem 4.2} Let \( N = 1, 2, \ldots \) be the horizon length. Consider the parameter \( \alpha^* \) defined in (4.1). If \( \alpha^* < 0 \), then Lemma 4.1 implies that Condition 3.3 holds. Therefore, the results follow from Theorem 3.4(i). On the other hand, if \( 0 \leq \alpha^* < 1 \), then Lemma 4.1 implies that Condition 3.3 does not hold.

(i) Suppose \( \alpha \in [0, \alpha^*] \). Lemmas 4.9(ii) and 4.10(ii) imply that \( G_{t, \alpha}(x), t = 0, 1, \ldots, N - 1 \), are convex functions and \( \lim_{x \to -\infty} G_{t, \alpha}(x) < +\infty \). Therefore, in view of Lemma 4.6, a policy that never orders at steps \( t = 0, 1, \ldots, N - 1 \) is optimal.

(ii) Suppose \( \alpha > \alpha^* \). Lemmas 4.6, 4.9(i), and 4.10(i) imply that (a) if \( N \leq N_\alpha \), then a policy that never orders at steps \( t = 0, 1, \ldots, N - 1 \) is optimal, and (b) if \( N > N_\alpha \), then the action \( a = 0 \) is always optimal at steps \( t = N - N_\alpha \), \( N - 1 \) and furthermore \( N_\alpha < +\infty \) and \( G_{N_\alpha, \alpha}(x) = f_{N_\alpha, \alpha}(x) \). In view of Lemma 4.7, the functions \( G_{t, \alpha}(x), t = N_\alpha, N_\alpha + 1, \ldots \), are \( K \)-convex and \( \lim_{|x| \to +\infty} G_{t, \alpha}(x) = +\infty \). These properties of the functions \( G_{t, \alpha}(x) \) imply the optimality of \( (s_t, S_t) \) policies at steps \( t = N - N_\alpha, \ldots, N - 1 \) described in statement (ii-b); see, e.g., the paragraph following the proof of Proposition 6.7 in Feinberg and Lewis (2015).

\[
\square
\]

\textbf{Proof of Theorem 4.4} Consider an infinite-horizon problem and the parameter \( \alpha^* \) defined in (4.1). If \( \alpha^* < 0 \), then Lemma 4.1 implies that Condition 3.3 holds. Therefore, statement (i) follows from Theorem 3.4(ii). On the other hand, if \( 0 \leq \alpha^* < 1 \), then Lemma 4.1 implies that Condition 3.3 does not hold.

(i) Suppose \( \alpha > \alpha^* \). Lemmas 4.9(i) and 4.10(i) imply that \( N_\alpha < +\infty \) and \( G_{N_\alpha, \alpha}(x) = f_{N_\alpha, \alpha}(x) \). Therefore, according to Lemma 4.8, the function \( G_\alpha(x) \) is \( K \)-convex and \( \lim_{|x| \to +\infty} G_\alpha(x) = +\infty \), and this implies statement (i).

(ii) Suppose \( \alpha \in [0, \alpha^*] \). According to Lemma 4.10(ii), the function \( G_\alpha(x) \) is convex and \( \lim_{x \to -\infty} G_\alpha(x) < +\infty \). Therefore, this function is nondecreasing. Therefore, \( G_\alpha(x) \leq K + G_\alpha(x + a) \) for all \( x \in \mathbb{X} \) and \( a \geq 0 \), and this implies that a policy that never orders is optimal.

\[
\square
\]

\textbf{5 Continuity of the value functions}

In this section we show that the value functions \( v_{N, \alpha}(x), N = 1, 2, \ldots \), and \( v_\alpha(x) \) are continuous in \( x \in \mathbb{X} \). As explained in Feinberg and Lewis (2015, Corollary 6.1), the general results on MDPs imply that these functions are inf-compact and therefore they are lower semi-continuous. As discussed above, these functions are \( K \)-convex. However, Example 5.1
illustrates that a $K$-convex function may not be continuous. Thus, the continuity of the
value functions $v_{N,\alpha}(x), N = 1, 2, \ldots,$ and $v_\alpha(x)$ follows neither from the known general
properties of value functions for infinite-state MDPs nor from these properties combined
with the $K$-convexity of these functions.

We recalled that a function $f : \mathbb{U} \to \mathbb{R} \cup \{+\infty\}$ for a metric space $\mathbb{U}$ is called lower semi-continuous, if the level set
\begin{equation}
\mathcal{D}_f(\lambda) := \{ u \in \mathbb{U} : f(u) \leq \lambda \},
\end{equation}
is closed for every $\lambda \in \mathbb{R}$. A function $f : \mathbb{U} \to \mathbb{R} \cup \{+\infty\}$ is called inf-compact, if the level set $\mathcal{D}_f(\lambda)$ defined in (5.1) is compact for every $\lambda \in \mathbb{R}$. Of course, each inf-compact function is lower semi-continuous. As proved in Feinberg and Lewis (2015) (see also Feinberg et al. 2012, Theorem 2), for an MDP with a standard Borel state space, if the one-step costs $c$ and weakly continuous transition probabilities; see Feinberg et al. (2013, Example 4.4).

For fixed $K > 0$ and $d \in [0, K]$, the following discontinuous function
\begin{equation*}
f(x) = \begin{cases} 
-x + K & \text{if } x < 0, \\
d & \text{if } x = 0, \\
x & \text{if } x > 0,
\end{cases}
\end{equation*}
is $K$-convex. To verify $K$-convexity, observe that this function is convex on $(-\infty, 0)$ and
$[0, +\infty)$. Let $x < 0$, $y \geq 0$, and $\lambda \in (0, 1)$. If $(1 - \lambda)x + \lambda y \neq 0$, then $f((1 - \lambda)x + 
\lambda y) \leq (1 - \lambda)f(x) + \lambda f(y) + \lambda K$. If $(1 - \lambda)x + \lambda y = 0$, then
\begin{equation*}
f(0) \leq K < (1 - \lambda)(-x) + \lambda y + K = (1 - \lambda)f(x) + \lambda f(y) + \lambda K.
\end{equation*}

The following theorem describes the continuity of value functions for finite-horizon
inventory control problems considered in this paper. The continuity of finite-horizon value
functions is proved by induction. From Theorem 4.2, we know that either $(s_1, S_1)$ policy or
a policy that does not order is optimal at epoch $t$. We prove that under these two cases the
value function $v_{t+1}$ is continuous if $v_t$ is a continuous function.

**Theorem 5.2** For a finite horizon inventory control problem, the functions $v_{t,\alpha}(x)$ and
$G_{t,\alpha}(x), t = 0, 1, \ldots,$ are continuous on $\mathbb{X}$ for all $\alpha \geq 0$.

**Proof** We prove by induction that the functions $v_{t,\alpha}(x)$ and $G_{t,\alpha}(x), t = 0, 1, \ldots,$ are continuous. Let $t = 0$. Then $v_{0,\alpha}(x) = 0$ and $G_{0,\alpha}(x) = \hat{c}x + \mathbb{E}[h(x - D)], x \in \mathbb{X}$. Therefore, for all $\alpha \geq 0$, the functions $v_{0,\alpha}(x)$ and $G_{0,\alpha}(x)$ are convex on $\mathbb{X}$ and hence they are continuous.

Now assume that $v_{t,\alpha}(x)$ and $G_{t,\alpha}(x)$ are continuous functions for some $t \geq 0$. According
to Theorem 4.2 and Lemma 4.10, one of the following cases takes place: (1) $G_{t,\alpha}(x)$ is a

\[ \square \text{ Springer} \]
convex function, \( \lim_{x \to -\infty} G_{t, \alpha}(x) < +\infty\), and the action \( a = 0 \) is optimal at all states when \( t \) periods are left or (2) \( G_{t, \alpha}(x) \) is a \( K \)-convex function, \( \lim_{x \to +\infty} G_{t, \alpha}(x) = +\infty\), and the \((s_{t, \alpha}, S_{t, \alpha})\) policy is optimal when \( t \) periods are left, where \( s_{t, \alpha} \) and \( S_{t, \alpha} \) are defined in (3.7) and (3.6) with \( f(x) := G_{t, \alpha}(x), x \in \mathbb{X} \).

Case (i) In view of (3.1), since an action that never orders is optimal, then \( v_{t+1, \alpha}(x) = G_{t, \alpha}(x) - \bar{c}x \). Therefore, convexity of the function \( G_{t, \alpha}(x) \) implies that \( v_{t+1, \alpha}(x) \) is a convex function. In view of (3.3), since \( h(x) \) and \( v_{t+1, \alpha}(x) \) are convex functions on \( \mathbb{X} \), then \( G_{t+1, \alpha}(x) \) is also a convex function. Since the functions \( v_{t+1, \alpha}(x) \) and \( G_{t+1, \alpha}(x) \) are convex on \( \mathbb{X} \). Thus they are continuous.

Case (ii) Since there exists an optimal \((s_{t, \alpha}, S_{t, \alpha})\) policy, then in view of (3.1), the function \( v_{t+1, \alpha}(x) \) can be written as

\[
v_{t+1, \alpha}(x) = \begin{cases} 
G_{t, \alpha}(x) - \bar{c}x & \text{if } x \geq s_{t, \alpha}, \\
K + G_{t, \alpha}(S_{t, \alpha}) - \bar{c}x & \text{if } x < s_{t, \alpha}.
\end{cases}
\]

(5.2)

In view of the definition of \( s_{t, \alpha} \) in (3.7) with \( f(x) := G_{t, \alpha}(x), x \in \mathbb{X} \), since the function \( G_{t, \alpha}(x) \) is continuous on \( \mathbb{X} \), then \( G_{t, \alpha}(s_{t, \alpha}) = K + G_{t, \alpha}(S_{t, \alpha}) \). Therefore, (5.2) implies that the function \( v_{t+1, \alpha}(x) \) is continuous.

Let us prove that the function \( G_{t+1, \alpha} \) is continuous. It is sufficient to prove that this function is continuous on each interval \((-\infty, b)\), where \( b \in \mathbb{R} \). Let us fix an arbitrary real number \( b \).

Let us consider the continuous function \( g_{t+1, \alpha}(x) = v_{t+1, \alpha}(x) + \bar{c}x \). This function is bounded on \((-\infty, b)\) because, in view of (5.2), \( g_{t+1, \alpha}(x) = K + G_{k, \alpha}(S_{t, \alpha}), \) when \( x < s_{t, \alpha} \). If \( x \in (-\infty, b) \) then \( x - D \in (-\infty, b) \) a.s. In addition, if \( x_n \to x \) then \( x_n - D \) converges weakly to \( x - D \). Since the function \( g_{t+1, \alpha} \) is bounded and continuous on \((-\infty, b)\), then the function \( E[g_{t+1, \alpha}(x - D)] \) is continuous on \((-\infty, b)\). Since \( b \) is arbitrary, this function is continuous on \( \mathbb{R} \). Formula (3.3) can be rewritten as

\[
G_{t+1, \alpha}(x) = (1-\alpha)\bar{c}x + E[h(x - D)] + \alpha E[g_{t+1, \alpha}(x - D)] + \alpha \bar{c}E[D],
\]

where all the summands are continuous functions. In particular, \( E[h(x - D)] \) is a nonnegative convex real-valued function on \( \mathbb{R} \), and therefore it is continuous. As shown above in this paragraph, the function \( E[g_{t+1, \alpha}(x - D)] \) is continuous too. Thus, the function \( G_{t+1, \alpha} \) is continuous. Hence, the induction arguments imply that \( v_{t, \alpha}(x) \) and \( G_{t, \alpha}(x), t = 0, 1, \ldots, \) are continuous functions.

Consider an infinite-horizon inventory control problem. If \((s_\alpha, S_\alpha)\) policy is optimal, then the value function can be written as follows.

\[
v_\alpha(x) = \begin{cases} 
G_\alpha(x) - \bar{c}x & \text{if } x \geq s_\alpha, \\
K + G_\alpha(S_\alpha) - \bar{c}x & \text{if } x < s_\alpha,
\end{cases}
\]

(5.3)

where \( G_\alpha(x) \) is defined in (3.4).

The following theorem describes the continuity of value functions for infinite-horizon inventory control problems considered in this paper. According to Feinberg et al. (2012, Theorem 2), we know that \( v_{N, \alpha}(x) \uparrow v_\alpha(x) \) as \( N \to +\infty \) for all \( x \in \mathbb{X} \). We further prove that such convergence is uniform, and the continuity of finite-horizon value functions implies the continuity of infinite-horizon value functions.

**Theorem 5.3** Consider an infinite-horizon inventory control problem with expected total discounted cost criterion. The functions \( v_\alpha(x) \) and \( G_\alpha(x) \) are continuous on \( \mathbb{X} \) for all \( \alpha \in [0, 1) \).
Proof Consider \( \alpha^* \) defined in (4.1). According to Theorem 4.2, if \( \alpha \in [0, \alpha^*] \), then a policy that never orders is optimal for infinite-horizon problem, which, in view of (3.2), implies that \( v_\alpha(x) = G_\alpha(x) - cx \). As follows from Lemma 4.10(ii), the function \( G_\alpha(x) \) is convex on \( \mathbb{X} \) and then continuous. Therefore the function \( v_\alpha(x) \) is also convex on \( \mathbb{X} \) and hence continuous.

Consider a \( N \)-horizon optimal policy \( \phi^N = (\phi_0^N, \phi_1^N, \ldots, \phi_{N-1}^N) \) and an infinite-horizon optimal policy \( \phi \). Define a policy \( \psi = (\psi_0, \psi_1, \ldots) \) as

\[
\psi_t = \begin{cases} 
\phi_t^N & \text{if } 0 \leq t \leq N - 1, \\
\phi & \text{if } t \geq N.
\end{cases}
\]

Then, for all \( x \in \mathbb{X} \),

\[
v_{N,\alpha}(x) \leq v_\alpha(x) \leq v_\psi^N(x) = v_{N,\alpha}(x) + \mathbb{E}_x \left[ \sum_{t=0}^{\infty} \alpha^t c(x_t, a_t) \right],
\]

where the first inequality holds because all costs are non-negative, the second inequality is straightforward, and the last equality follows from the definition of \( v_\psi^N(x) \) and the optimality of \( \phi^N \).

If \( \alpha \in (\alpha^*, 1) \), then \( \phi \) can be chosen as the \((s_\alpha, S_\alpha)\) policy; Theorem 4.2. Consider \( N > N_\alpha \) whose existence is stated in Theorem 4.2(ii), and \( s_{t,\alpha} \) and \( s_{t,\alpha} \) defined in (3.6) and (3.7) respectively with \( f(x) := G_{t,\alpha}(x) \), \( x \in \mathbb{X} \) for all \( t = N_\alpha, \ldots, N - 1 \). Then Theorem 4.2(ii) implies that \( \phi^N \) can be chosen as the policy that follows \((s_{N-t-1,\alpha}, S_{N-t-1,\alpha})\) policy, \( t = 0, \ldots, N - N_\alpha - 1 \), and from then on never orders the inventory.

Let us fix \( z \in \mathbb{X} \). Consider \( x_0 \in (-\infty, z) \). According to Feinberg and Lewis (2015, Theorem 6.11(ii)), each sequence of pairs \( \{(s_{t,\alpha}, S_{t,\alpha})\}_{t=N_\alpha,N_\alpha+1,\ldots} \) is bounded. Thus there exists a constant \( M_S \) such that \( s_{t,\alpha} \leq M_S \) for all \( t = 0, 1, \ldots \).

Since \( \psi_t = \phi_t^N \) for all \( t \leq N - 1 \), then given \( x_0 < z \),

\[
s_{N_\alpha} - S_{N_\alpha} \leq x_N \leq \max\{x_0, S_{N_\alpha}, \ldots, S_{N-1}\} \leq \max\{z, M_S\}, \quad a.s.,
\]

where the first two inequalities hold because \( \phi^N \) is \((s, S)\) policy at first \( N - N_\alpha \) steps and never orders at the remaining \( N_\alpha \) steps, and the last inequality holds because \( x_0 < z \) and \( s_{t,\alpha} \leq M_S \) for all \( t = 0, 1, \ldots \).

Since \( \psi_t = \phi \) for all \( t \geq N \), then in view of (5.5) and the definition of \( \phi \), the following inequalities hold:

\[
0 \leq a_t \leq \max\{0, S_{\alpha} - x_t\}, \quad t \geq N, \quad (5.6)
\]

\[
s_{\alpha} - D \leq x_t \quad a.s., \quad t > N, \quad (5.7)
\]

\[
s_{\alpha} \leq x_t + a_t \leq \max\{x_N, S_{\alpha}\} \leq \max\{z, M_S, S_{\alpha}\}, \quad t \geq N. \quad (5.8)
\]

According to Theorem 4.2, \( s_{\alpha} \) and \( S_{\alpha} \) are real numbers. Therefore, (5.8) implies that there exists a constant \( M_1 \) such that for all \( t \geq N \),

\[
\mathbb{E}[h(x_t + a_t - D)] \leq \mathbb{E}[h(s_{\alpha} - D)] + \mathbb{E}[h(\max\{z, M_S, S_{\alpha}\} - D)] \leq M_1, \quad (5.9)
\]

where the first inequality holds because the function \( \mathbb{E}[h(x - D)] \) is non-negative and convex on \( \mathbb{X} \), and the last one holds because \( z, M_S, \) and \( S_{\alpha} \) are real numbers and \( \mathbb{E}[h(x - D)] \) is \( +\infty \), \( x \in \mathbb{X} \).

In addition, there exists a constant \( M_2 \) such that for all \( t \geq N \) and \( x_0 < z \),

\[
0 \leq \mathbb{E}^S_{x_0}[a_t] \leq \mathbb{E}[\max\{0, S_{\alpha} - s_\alpha + D, S_{\alpha} - S_{N_\alpha} + S_{N_\alpha+1}\}] = \max\{0, S_{\alpha} - s_\alpha + \mathbb{E}[D], S_{\alpha} - S_{N_\alpha} + (N_\alpha + 1)\mathbb{E}[D]\} \leq M_2, \quad (5.10)
\]
where the first two inequalities follows from (5.6), (5.7) and (5.5), the equality is straightforward, and the last inequality holds because all quantities are real numbers.

In view of (2.4), (5.9) and (5.10), for $M_3 := K + M_1 + M_2$ such that for all $t \geq N$ and $x_0 < z$,
\begin{equation}
\mathbb{E}_{x_0}^\psi[c(x_t, a_t)] \leq K + c\mathbb{E}_{x_0}^\psi[a_t] + \mathbb{E}[h(x_t + a_t - D)] \leq M_3 < +\infty. \tag{5.11}
\end{equation}

Since the cost $c(x, a)$ is non-negative for all $(x, a) \in \mathbb{X} \times A$, then according to Feinberg et al. (2012, Theorem 2), $v_{N, \alpha}(x) \uparrow v_{\alpha}(x)$ as $N \to +\infty$ for all $x \in \mathbb{X}$. Therefore,
\begin{equation}
\sup_{x_0 \in (-\infty, z)} |v_{\alpha}(x_0) - v_{N, \alpha}(x_0)| \leq \mathbb{E}_{x_0}^\psi \sum_{t = N}^\infty \alpha^t c(x_t, a_t)],
\end{equation}
where the first inequality follows from (5.4), the second one follows from (5.11), and the equality is straightforward. In view of (5.12), the function $v_{N, \alpha}(x)$ converges uniformly to the function $v_{\alpha}(x)$ on $(-\infty, z)$ as $N \to +\infty$. Therefore, according to the uniform limit theorem, since the function $v_{N, \alpha}(x)$ is continuous on $\mathbb{X}$ for all $N = 1, 2, \ldots$ (Theorem 5.2), then the function $v_{\alpha}(x)$ is continuous on $(-\infty, z)$. Since $z$ can be chosen arbitrarily, thus the function $v_{\alpha}(x)$ is continuous on $\mathbb{X}$.

Let us fix $y \in \mathbb{X}$. Define the following function
\[ g_{\alpha}(x) = \begin{cases} v_{\alpha}(x) + \bar{c}x & \text{if } x \leq y + 1, \\ v_{\alpha}(y + 1) + \bar{c}(y + 1) & \text{if } x > y + 1. \end{cases} \]
Since the functions $v_{\alpha}(x)$ and $\bar{c}x$ are continuous, then the function $g_{\alpha}(x)$ is continuous. In view of (3.2), the function $g_{\alpha}(x)$ is bounded on $\mathbb{X}$. Therefore,
\begin{equation}
\lim_{z \to y} \{(1 - \alpha)\bar{c}z + E[h(z - D)] + \alpha E[g_{\alpha}(z - D)]\}
\end{equation}
\begin{equation}
= (1 - \alpha)\bar{c}y + E[h(y - D)] + \alpha E[g_{\alpha}(y - D)],
\end{equation}
where the equality holds since the function $\bar{c}x$ is continuous, the function $E[h(x - D)]$ is convex on $\mathbb{X}$ and hence it is continuous, and $z - D$ converges weakly to $y - D$ as $z \to y$ and the function $g_{\alpha}(x)$ is continuous and bounded.

Observe that $G_{\alpha}(x) = (1 - \alpha)\bar{c}x + E[h(x - D)] + \alpha E[g_{\alpha}(x - D)] + \alpha \bar{c}E[D]$ for all $x \leq y + 1$, then (5.13) implies that $\lim_{z \to y} G_{\alpha}(z) = G_{\alpha}(y)$. Therefore, the function $G_{\alpha}(x)$ is continuous.

Theorems 5.2 and 5.3 imply the following corollary.

**Corollary 5.4** Let $\mathbb{X} = \mathbb{R}$ and $A = \mathbb{R}_+$. The statements of Theorems 3.4, 4.2, and 4.4 remain correct, if the second sentence of Definition 3.2 is modified in the following way: a policy is called an $(s_t, S_t)$ policy at step $t$, if it orders up to the level $S_t$, if $x_t < s_t$, does not order, if $x_t > s_t$, and either does not order or orders up to the level $S_t$, if $x_t = s_t$.

**Proof** The proofs of Theorems 3.4, 4.2, and 4.4 are based on the fact that $K + g(S) < g(x)$, if $x < s$, and $K + g(S) \geq g(x)$, if $x \geq s$, where $g = G_{t, \alpha}, S = S_{t, \alpha}$, and $s = s_{t, \alpha}$ for a finite-horizon problem and $g = G_{\alpha}, S = S_{\alpha},$ and $s = s_{\alpha}$ for the infinite-horizon problem. Since the function $g$ is continuous in both cases, we have that $K + g(S) = g(s)$. Thus both actions are optimal at the state $s$. 

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Remark 5.5 Corollary 5.4 also follows from the properties of the sets of optimal decisions $A_{t,\alpha}(x) := \{a \in A : v_{t+1,a}(x) = c(x, a) + \alpha E[v_{t,a}(x + a - D)], \ t = 0, 1, \ldots, \}$ for finite-horizon problems and $A_{\alpha}(x) := \{a \in A : v_0(x) = c(x, a) + \alpha E[v_0(x + a - D)]\}$ for infinite-horizon problem, $x \in X$, where the one-step cost function $c$ is defined in (2.4). The solution multifunctions $A_{t,\alpha}(\cdot), \ t = 0, 1, \ldots, \$ and $A_{\alpha}(\cdot)$ are compact-valued (see Feinberg et al. 2012, Theorem 2; Feinberg and Lewis 2015, Theorem 3.4) and upper semi-continuous (this is true in view of Feinberg and Kasyanov 2015, Statement B3 (see also Feinberg et al. 2014, p. 1045) because the value functions are continuous and the optimality operators take infimums of inf-compact functions). Since upper semi-continuous, compact-valued set-valued functions are closed (Nikaido 1968, Lemma 4.4), the graphs of the solution multifunctions are closed. Since $0 \in \bar{A}_0(x)$ for all $x > s$, then $0 \in A(s)$. Similarly, $0 \in A_{t,\alpha}(s_t), \ t = 0, 1, \ldots$.

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References


