OPTIMALITY OF RANDOMIZED TRUNK RESERVATION FOR A PROBLEM WITH MULTIPLE CONSTRAINTS

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Abstract

We study optimal admission of arriving customers to a Markovian finite-capacity queue, e.g. M/M/c/N queue, with several customer types. The system managers are paid for serving customers and penalized for rejecting them. The rewards and penalties depend on customer types. The penalties are modelled by a K-dimensional cost vector, $K \geq 1$. The goal is to maximize the average rewards per unit time subject to the K constraints on the average costs per unit time. Let K_m denote $min\{K, m-1\}$, where m is the number of customer types. For a feasible problem, we show the existence of a K_m-randomized trunk reservation optimal policy, where the acceptance thresholds for different customer types are ordered according to a linear combination of the service rewards and rejection costs. In addition, we prove that any K_m-randomized stationary optimal policy has this structure.

1 Introduction and Problem Formulation

In this paper, we consider a controlled finite capacity Markovian queue with $m = 1, 2, \ldots$ types of customers arriving according to independent Poisson processes with the intensities λ_i, $i = 1, \ldots, m$, respectively. When a customer arrives, its type becomes known. When there are N customers in the system, the system is full and new arrivals are lost. If the system is not full, upon an arrival of a new customer, a decision of accepting or rejecting this customer is made. A positive reward r_i is collected upon completion of serving an accepted type i customer. A nonnegative cost vector $C_i = (C_{1,i}, C_{2,i}, \ldots, C_{K,i})^\tau$ incurs due to the rejection or lost of an arriving type i customer, where K is the number of constraints in this problem. The service time of a customer does not depend on the customer type. When there are n customers in the queue, the departure rate is μ_n, $n = 1, \ldots, N$. The numbers μ_n, $n = 1, \ldots, N$, satisfy the condition $\mu_{n-1} \leq \mu_n$, where $\mu_0 = 0$ and $\mu_1 > 0$. In particular, for an $M/M/c/N$ queue, for some $\mu > 0$,

$$\mu_i = \begin{cases} i\mu, & \text{if } i = 1, \ldots, c, \\ c\mu, & \text{if } i = c+1, \ldots, N. \end{cases}$$

Unless otherwise specified, we do not assume that $r_1 \geq r_2 \geq \cdots \geq r_m$.

Our goal is to maximize the average rewards per unit time, subject to multiple constraints on average costs per unit time. This research is motivated by the following question: what is the structure of optimal policies for the problem when the blocking probabilities for some of the customer types do not exceed given numbers? The answer to this question is given in Corollary 4.3 below. Previously Fan-Orzechowski and Feinberg [3] solved such a problem with a single constraint on the blocking probability of one type of customers. Feinberg and Reiman [8] solved a more particular problem when all the rewards are different and the constraint on the blocking probability is applied to the most profitable type of customers.

Consider $\mathcal{K} = 0, 1, \ldots, m-1$. A $\mathcal{K}$-randomized trunk reservation policy ϕ is defined by m numbers M_i^ϕ, $0 \leq M_i^\phi \leq N-1$, $i = 1, \ldots, m$. Among these numbers $M_1^\phi, \ldots, M_m^\phi$, at most $\mathcal{K}$ numbers are non-integer and at least one number equals $N-1$. For a number M we denote by $\lfloor M \rfloor$ the integer part of M. If the system is controlled by the policy ϕ, a type i arrival will be admitted with probability 1 if it sees no more than $\lfloor M_i^\phi \rfloor$ customers in the system, it will be rejected if the number of customers it sees in the system exceeds $\lfloor M_i^\phi \rfloor + 1$, and it will be accepted with the probability $(M_i^\phi - \lfloor M_i^\phi \rfloor)$ if there are exactly $\lfloor M_i^\phi \rfloor + 1$ customers in the system prior to the time of its arrival. In particular, if the number M_i^ϕ is an integer, a type i arrival will be admitted if and only if it sees no more that M_i^ϕ customers in the system. Thus, $M_i^\phi = N-1$ means that a type i arrival is admitted whenever the system is not full. A randomized trunk reservation policy ϕ is called consistent with a function r' defined on the set $\{1, \ldots, m\}$ if $r'_i > r'_j$ implies $M_i^\phi \geq M_j^\phi$ for $i, j = 1, \ldots, m$. If all the thresholds are integer, the randomized trunk reservation policy is called a trunk reservation policy. We sometimes write M_i instead of M_i^ϕ for the thresholds when there is only one policy in the context and no confusion will occur.

In this paper we consider a more general setting where the number of constraints K is not necessarily less than m and introduce $K_m = min\{K, m-1\}$. We prove that, if the problem is feasible, there exists a K_m-randomized trunk reservation policy which is consistent with the reward function

$$r'_i = r_i + \sum_{k=1}^{K} \bar{u}_k C_{k,i}, \quad i = 1, \ldots, m, \tag{1.1}$$

where $\bar{u}_k \geq 0$ is the Lagrangian multiplier with respect to the k-th constraint of the linear programming problem formulated in this paper. In addition, Theorem 4.1 shows that any K_m-randomized stationary optimal policy is a K_m-randomized trunk reservation policy consistent with r'.

In Feinberg and Reiman [8, Sections 6 and 7], several more predictable optimal policies and optimal non-randomized strategies were constructed. Similar results can be obtained for the more general problem considered in this paper. In fact, these constructions hold as long as the optimality of randomized trunk reservation policies is established.

Miller [15] studied a one-criterion problem for an $M/M/c/loss$ queue when $r_1 > r_2 > \cdots > r_m$. In this case, there exists an optimal non-randomized trunk reservation policy which is consistent with r. In other words, all the thresholds M_i are integers and $N - 1 = M_1 \geq M_2 \geq \cdots \geq M_m$. Feinberg and Reiman [8] studied a constrained problem with $r_1 > r_2 > \cdots > r_m$ where the goal is to maximize average rewards per unit time subject to the constraint that the blocking probability for type 1 customers does not exceed a given level. Feinberg and Reiman [8] proved the existence of an optimal 1-randomized trunk reservation policy with $N-1 = M_1 \geq M_2 \geq \cdots \geq M_m$. Fan-Orzechowski and Feinberg [3] considered a problem with a single constraint and the goal is to maximize average rewards per unit time subject to the constraint on average costs. They proved the existence of a 1-randomized trunk reservation policy which is consistent with the reward function

$$r_i' = r_i + \bar{u}_1 c_i,$$

where $\bar{u}_1 \geq 0$ is the Lagrangian multiplier with respect to the first constraint of the linear programming problem formulated in that paper. In particular, Fan-Orzechowski and Feinberg [3] solved the problem with one constraint on the blocking probability for type k customers, $k = 1, \ldots, m$.

In addition to the classical Miller's [15] problem formulation, various versions and generalizations of the admission problem have been studied in the literature. The references could be found in Fan-Orzechowski and Feinberg [3]. Recent research in this area includes Lewis et al. [10, 11], Lewis [9], Lin and Ross [12, 13], Altman et al. [1], and Altman [2]. If service times depend on customer types or different types of customers require different numbers of servers, the problem becomes NP-hard and trunk reservation may not be optimal; see Ross [16, p.137] and Altman et al. [1].

This paper is organized as follows. We formulate the problem as a Semi-Markov Decision Process (SMDP) and give preliminary results in Section 2. In section 3 we give the formulation of a linear program (LP) that identifies an optimal policy and introduce Lagrangian relaxation to reduce the number of constraints. In section 4, the proof of the main theorem and its applications are presented.

2 Semi-Markov Decision Model and Preliminary Results

Following Feinberg and Reiman [8], and Fan-Orzechowski and Feinberg [3], we model the problem via a semi-Markov decision process (SMDP). Since the sojourn time between actions are exponential distributed, this problem is an exponential semi-Markov decision process (ESMDP); please refer to Feinberg [7] for more details. Notice that this problem can also be formulated as a continuous time Markov decision process (CMDP). The extra technical difficulty in using CMDP is to prove that the controlled process has no absorbing states; see Feinberg [6]. We then can reach the same preliminary result as by using the SMDP model in this paper: when the problem is feasible, there exists a randomized stationary optimal policy that uses a randomization procedure in at most K_m states.

In the framework of an SMDP model, we define the state space $I = \{0, 1, \ldots, N-1\} \bigcup (\{0, 1, \ldots, N\} \times \{1, \ldots, m\})$, which represents the departure and arrival epochs. If the state of the system is $n = 0, \ldots, N-1$, it means that a departing customer leaves n customers in the system. The state (n, i) means that an arrival of type i sees n customers in the system.

The action set $A = \{0, 1\}$. For $n = 0, \ldots, N-1$ and $i = 1, \ldots, m$, we set $A(n, i) = A = \{0, 1\}$ and $A(N, i) = \{0\}$, where the action 0 means that the type i arrival should be rejected or is lost and the action 1 means that it should be accepted. In any state $n = 0 \ldots, N-1$, we set $A(n) = \{0\}$. These are departure epoches and the decision maker does not decide to accept or reject customers in these states. Therefore, we model these actions sets $A(n)$ as singletons. The definitions of the sojourn time and transition probabilities are presented in [3, section 2] and we do not repeat them here.

For simplicity, let the reward be collected when an arrival is accepted. Therefore,

$$r(s,a)=\begin{cases} r_i, & \text{if} \quad s=(n,i), n=0,\dots,N-1, \text{and} \quad a=1;\\ 0, & \text{otherwise,}\end{cases}$$

and for $k=1,\dots,K$,

$$c_k(s,a)=\begin{cases} C_{k,i}, & \text{if} \quad s=(n,i), n=0,\dots,N, \text{and} \quad a=0;\\ 0, & \text{otherwise.}\end{cases}$$

We define the long-run average rewards earned by the system as

$$W_0(z,\pi)=\liminf_{t\to\infty} t^{-1}\mathbb{E}_z^\pi \sum_{n=0}^{N(t)-1} r(x_n,a_n),$$

and the long-run average costs of the system as

$$W_k(z,\pi)=\limsup_{t\to\infty} t^{-1}\mathbb{E}_z^\pi \sum_{n=0}^{N(t)-1} c_k(x_n,a_n), \quad k=1,\dots,K,$$

where z is an initial state, π is a strategy, x_n is the state at epoch t_n, $\mathbb{E}_z^\pi$ is the expectation operator for the initial state z and the strategy π, and $N(t)=\max\{n: t_n\le t\}$ is the number of jumps by time epoch t.

A strategy is called a randomized stationary policy if assigned actions a_n depend only on the current state x_n. In addition, if a_n is a deterministic function of x_n, the corresponding strategy is called a stationary policy. A randomized stationary policy ϕ is called k-randomized stationary, $k=0,1,\dots,$ if the number of states (n,i) such that $0<\phi(n,i)<1$ is less than or equal to k. The notions of stationary and 0-randomized stationary policies coincide.

Notice that the Unichain Condition, any stationary policy defines a Markov Chain with one recurrent class, holds for this model. If an SMDP satisfies the Unichain Condition then, according to Feinberg [5, Theorem 9.2], for a feasible problem with average rewards per unit time and K constraints, there exists an optimal K-randomized stationary policy. Therefore, similarly to Fan-Orzechowski and Feinberg [3], we model our problem as:

$$\begin{aligned} &\text{maximize} \quad W_0(\phi)\\ &\text{subject to} \quad W_k(\phi)\le G_k, \quad k=1,\dots,K, \end{aligned} \tag{2.1}$$

with a randomized stationary policy ϕ as the variable. Since the action sets are non-singletons only at the arrival epochs, a randomized stationary policy ϕ for our problem is defined by $\phi(n,i)$, $n=0,\dots,N-1$, $i=1,\dots,m$, the probability of accepting an arrival of type i when the arrival sees n customers in the system.

Similar to [3], we consider the following unconstrained problem:

$$\text{maximize} \quad W(\phi). \tag{2.2}$$

Note that we do not assume distinct rewards among different customer types. This extension may not seem significant at first glance but it is important. In fact, even if we assume that all rewards r_i are distinct, after the

Lagrangian relaxation, it is possible that $r'_i = r'_j$ for some $i, j = 1, \ldots, m$, in a new unconstrained problem; see Section 3, where the Lagrangian relaxation is introduced and the connection between an unconstrained problem and a constrained problem is established in Lemma 3.1.

The following lemma is used in the proof of Theorem 3.3 and Theorem 4.1.

Lemma 2.1. *[3, Lemma 3.3] Consider any randomized stationary optimal policy ϕ for the unconstrained problem (2.2). (i) For any i, j, such that $r_i > r_j$,*

$$\phi(n,i) \geq \phi(n,j), n = 0, \ldots, N-1, \; i,j = 1, \ldots, m. \tag{2.3}$$

(ii) For each $n = 0, \ldots, N-1$, if there exist two customer types j_1 and j_2 such that $0 < \phi(n,j) < 1$, $j = j_1, j_2$, then $r_{j_1} = r_{j_2}$. In particular, if all the rewards $r_1, \ldots, r_m$ are different, then for each $n = 0, \ldots, N-1$ all the probabilities $\phi(n,j)$, $j = 1, \ldots, m$, except at most one, are equal to either 0 or 1.

(iii) There exists at least one customer type, say type ℓ, such that

$$\phi(n,\ell) = 1, n = 0, \ldots, N-1. \tag{2.4}$$

In particular, if $r_j = \max\{r_i | \, i = 1, \ldots, m\}$ then (2.4) holds with $\ell = j$.

(iv)

$$\phi(n,j) \geq \phi(n+1,j), n = 0, \ldots, N-2, j = 1, \ldots, m, \tag{2.5}$$

and for each $j = 1, \ldots, m$, all the probabilities $\phi(n,j)$, $n = 0, \ldots, N-1$, except at most one, are equal to either 0 or 1.

Corollary 2.2. *Any randomized stationary optimal policy ϕ for the unconstrained problem (2.2) is an $(m-1)$-randomized trunk reservation policy consistent with the rewards r_i. In addition, if all the rewards r_i are distinct, such a policy is s-randomized, where $s = min\{m-1, N\}$.*

Corollary 2.3. *Any stationary optimal policy ϕ for the unconstrained problem (2.2) is a trunk reservation policy consistent with the rewards r_i.*

3 Linear Programming Formulation and Lagrangian Relaxation

Consider the following Linear Program (LP) with variables (x, P), where $x = \{x(n,i) : \; n = 0, \ldots, N-1, i = 1, \ldots, m\}$, $P = (P_0, \ldots, P_N)$.

$$\text{maximize}_{x,P} \quad \sum_{i=1}^{m} \lambda_i r_i \sum_{n=0}^{N-1} x(n,i) \tag{3.1}$$

$$\text{subject to} \quad \sum_{i=1}^{m} \lambda_i C_{k,i} (1 - \sum_{n=0}^{N-1} x(n,i)) \leq G_k, \quad k = 1, \ldots, K, \tag{3.2}$$

$$\sum_{i=1}^{m} \lambda_i x(n,i) = \mu_{n+1} P_{n+1}, \quad n = 0, 1, \ldots, N-1, \tag{3.3}$$

$$\sum_{n=0}^{N} P_n = 1, \tag{3.4}$$

$$0 \leq x(n,i) \leq P_n, \quad n = 0, \ldots, N-1, \; i = 1, \ldots, m. \tag{3.5}$$

In view of (3.4) and (3.5), the feasible region of LP (3.1)-(3.5) is bounded. Therefore, this LP has an optimal solution, if it is feasible. If LP (3.1)-(3.5) is feasible, we consider an arbitrary optimal dual solution $(\bar{u}, \bar{v})$, $\bar{u} = (\bar{u}_1, \ldots, \bar{u}_{2mN+K})$ and $\bar{v} = (\bar{v}_1, \ldots, \bar{v}_{N+1})$, where $\bar{u}$ corresponds to the inequality constraints and $\bar{v}$ corresponds to the equality constraints, and introduce the following LP:

$$
\begin{aligned}
\text{maximize}_{x,P} \quad & \sum_{i=1}^{m} \lambda_i (r_i + \sum_{k=1}^{K} \bar{u}_k C_{k,i}) \sum_{n=0}^{N-1} x(n,i) - \sum_{k=1}^{K} \bar{u}_k (\sum_{i=1}^{m} \lambda_i C_{k,i} - G_k) \\
\text{subject to} \quad & (3.3) - (3.5).
\end{aligned}
\tag{3.6}
$$

Here $\bar{u}_k$ is also called the Lagrangian multiplier with respect to the k-th constraint in the prime LP (3.1)-(3.5). More details about the Lagrangian function and multipliers can be found in Appendix A. Lemma A.1 and explanations preceding it imply the following result.

Lemma 3.1. *If the LP (3.1)-(3.5) is feasible then: (i) any optimal solution of the LP (3.1)-(3.5) is an optimal solution of the LP (3.6), and (ii) the optimal values of objective functions for these two LPs are equal.*

This lemma plays an important role in the proof of our main theorem, Theorem 4.1. We refer to this technique as the Lagrangian relaxation in our paper. In general, the Lagrangian relaxation refers to using weak duality theorem in obtaining lower bounds for non-linear programming problems. We use the term "relaxation" here in the sense that, although the optimal set is enlarged after the transformation, the optimal value remains the same.

For a vector (x, P) satisfying (3.2)-(3.5), consider a randomized stationary policy ϕ such that:

$$
\phi(n,i) = \begin{cases} x(n,i)/P_n, & \text{if } P_n > 0,\ n = 0, 1, \ldots, N-1,\ i = 1, 2, \ldots, m; \\ \text{arbitray}, & \text{otherwise.} \end{cases} \tag{3.7}
$$

Lemma 3.2. *[3, Corollary 2.1] (i) If (x, P) is an optimal solution of LP (3.1), (3.3)-(3.5) then $P_n > 0$ for all $n = 0, 1, \ldots, N$.*

(ii) A randomized stationary policy ϕ is optimal for the problem (2.2) if and only if

$$
\phi(n,i) = x(n,i)/P_n, \qquad n = 0, 1, \ldots, N-1,\ i = 1, 2, \ldots, m, \tag{3.8}
$$

holds for an optimal solution (x, P) of LP (3.1), (3.3)-(3.5). In addition, if (x, P) is a basic optimal solution of LP (3.1), (3.3)-(3.5), then the policy ϕ defined in (3.8) is (non-randomized) stationary optimal.

Theorem 3.3. *(i) A randomized stationary policy ϕ is feasible for the problem (2.1) if and only if (3.7) holds for a feasible vector (x, P) of LP (3.1)-(3.5).*

(ii) If (x, P) is an optimal solution of LP (3.1)-(3.5) then $P_n > 0$ for all $n = 0, 1, \ldots, N$.

(iii) A randomized stationary policy ϕ is optimal for problem (2.1) if and only if Eq. (3.8) holds for an optimal solution (x, P) of LP (3.1)-(3.5). In addition, if (x, P) is a basic optimal solution of LP (3.1)-(3.5), then the policy ϕ defined by (3.8) is K_m-randomized stationary optimal.

Proof. The proof of (i),(ii), and the first statement of (iii) are identical to the proof of Theorem 2.1 in [3]. We shall prove the second statement of (iii). First, consider the case when $K < m$. We represent LP (3.1)-(3.5) in a standard LP form, where non-negative variables S_k, $k = 1, \ldots, K$, are introduced to replace (3.2) with $\sum_{i=1}^{m} \lambda_i C_{k,i}(1 - \sum_{n=0}^{N-1} x(n,i)) + S_k = G_k$ and non-negative variables $y(n,i)$, $n = 0, \ldots, N-1$, $i = 1, \ldots, m$, are introduced to replace (3.5) with $x(n,i) + y(n,i) = P_n$. There are $K + N + 1 + N \times m$ constraints and $K + 1 + N + 2(N \times m)$ variables for this new LP. Therefore, any basic optimal solution of this new LP has at most $K + N + 1 + N \times m$ basic variables. Since P_n, $n = 0, \ldots, N$, are positive, there are at most $K + N \times m$ basic variables among $x(n,i)$ and $y(n,i)$. Because $x(n,i) + y(n,i) = P_n > 0$, $x(n,i)$ and $y(n,i)$ cannot be equal to zero simultaneously. Therefore, for each pair (n,i), either $x(n,i) = 0$ or $y(n,i) = 0$, except at most K pairs where both $x(n,i)$ and $y(n,i)$ are not equal to zero. Since $\phi(n,i) = x(n,i)/P_n$, we have that for all pairs (n,i), except at most K, $\phi(n,i)$ equals either 0 or 1. Therefore, the policy ϕ is K-randomized stationary optimal. For $K \geq m$, we note that the matrix $C = (C_{k,i})$ is $K \times m$ for constraints in (3.2) and thus, its rank is at most m. After removing redundant constraints in (3.2), if there are l constraints left in (3.2) and $l < m$, we are back to the previous case and the proof is completed. If $l = m$, the only extra piece we need to add is to show that the policy ϕ is $(m-1)$-randomized. Lemma 3.1 and 3.2 imply that ϕ is optimal for an unconstrained problem. The rest follows from Lemma 2.1 (iii). ∎

We remark that, in the case of $K \geq m$, it is also intuitive that the randomized trunk reservation policy actually has at most $m - 1$ randomized entries. Indeed, for the most profitable customer type, who has the highest reward r', we will always accept it when the system is not full. Otherwise the system will be idling and waiting to serve the next less profitable customer, thus, the policy is suboptimal. This is also consistent with the definition of a $\mathcal{K}$-randomized trunk reservation, in which at least one threshold equals $N - 1$.

4 Main Theorem and Its Applications

We recall that $K_m = min\{K, m-1\}$.

Theorem 4.1. *Any K_m-randomized stationary optimal policy for problem (2.1) is a K_m-randomized trunk reservation policy, which is consistent with the reward function $r'_i = r_i + \bar{u}^K C_i$, $i = 1, \ldots, m$, where $\bar{u}^K = (\bar{u}_1, \bar{u}_2, \ldots, \bar{u}_K) \geq 0$ is a vector of Lagrangian multipliers with respect to the first K constraints in LP (3.1)-(3.5).*

Proof. Consider any K_m-randomized stationary optimal policy ϕ for problem (2.1). Corollary 3.2 (ii), Theorem 3.3 (iii), and Lemma 3.1 imply that ϕ is optimal for an unconstrained problem with the rewards $r'_i = r_i + \sum_{k=1}^{K} \bar{u}_k C_{k,i}$. Lemma 2.1 implies that ϕ is a randomized trunk reservation policy consistent with $r'_i = r_i + \sum_{k=1}^{K} \bar{u}_k C_{k,i}$. ∎

Consider an average reward SMDP with K constraints. If the Unichain Condition holds and a feasible policy exists, then there exists a K-randomized stationary optimal policy; see [5]. It is well known that for any LP which has at least one optimal solution, there exist an optimal basic solution. Therefore, Theorem 3.3 (iii) implies the existence of a K_m-randomized stationary optimal policy.

Corollary 4.2. *If the problem (2.1) is feasible, then there exists an optimal K_m-randomized trunk reservation policy which is consistent with the reward function r' defined in Theorem 4.1.*

According to [8, p. 471], for the costs vector C_i defined by

$$C_{k,i} = \begin{cases} {\lambda_k}^{-1}, & \text{if } i,k = 1,\dots,m \text{ and } i = k; \\ 0, & \text{otherwise,} \end{cases} \tag{4.1}$$

the average cost $W_k(z,\pi)$ is the blocking probability for type k customers. In addition, as mentioned before, since there exists at least one type of the most profitable customers in terms of rewards r', the system will always accept this customer type as long as there is empty seat to maximize the average rewards.

Corollary 4.3. *Consider a special case of the problem (2.1) with K constraints on the blocking probability of K types of customers, $K \le m$. Denote these customer types by J; $|J| = K$. If this problem is feasible then any K_m-randomized stationary optimal policy is K_m-randomized trunk reservation policy consistent with the reward function r' defined as*

$$r'_i = \begin{cases} r_i + \bar{u}_i/\lambda_i, & i \in J; \\ r_i, & \text{otherwise,} \end{cases}$$

where $\bar{u}^{K_m} = (\bar{u}_1, \bar{u}_2, \dots, \bar{u}_{K_m}) \ge 0$ is the vector of Lagrangian multipliers with respect to the first K constraints in LP (3.1)-(3.5), and

$$C_{k,i} = \begin{cases} {\lambda_k}^{-1}, & \text{if } i,k = 1,\dots,m, i \in J, \text{ and } i = k; \\ 0, & \text{otherwise.} \end{cases} \tag{4.2}$$

Therefore, there exists a K_m-randomized trunk reservation optimal policy consistent with the reward r' if the problem is feasible.

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A Appendix: Lagrangian Relaxation

In this appendix we present the results on Lagrangian optimization in convex programming and linear programming used in this paper. Let us consider a mathematical programming problem P:

$$\begin{array}{rrcll} \text{minimize} & f(x) & & & \\ \text{subject to} & g_i(x) & \leq & 0, & i=1,\dots,s, \\ & h_i(x) & = & 0, & i=1,\dots,p, \\ & x & \in & \mathbb{R}^n. & \end{array}$$

The problem P is a convex programming problem if (i) f and $g_1, \ldots, g_s$ are convex functions and (ii) $h_1, \ldots, h_p$ are linear functions. If all the functions f, g_i, $i = 1, \ldots, s$, and h_i, $i = 1, \ldots, p$ are linear, problem P becomes a linear programming problem.

The set $S = \{x \in \mathbb{R}^n : g_i(x) \le 0, i = 1, \ldots, s; h_i(x) = 0, i = 1, \ldots, p\}$ is called a feasible set and any $x \in S$ is called a feasible point. We consider the vector-valued functions $g = (g_1, \ldots, g_s)^\tau$ and $h = (h_1, \ldots, h_p)^\tau$. For a vector-valued function $F(x) = (F_1(x), \ldots, F_N(x))^\tau$ defined for some $x \in \mathbb{R}^n$, where $N = 1, 2, \ldots$, we denote by $\nabla F(x)$ the $N \times n$ gradient matrix with the elements $\partial F_i(x)/\partial x_i$ whenever all these partial derivatives exist at the point $x = (x_1, \ldots, x_n)$. The following two statements are well-known [14, p. 201]: (i) if $\nabla F(x)$ exists at x and ∇F is continuous at x, then F is differentiable at x, and (ii) if F is differentiable at x, it is continuous at x and $\nabla F(x)$ exists. Define two row vectors $u = (u_1, \ldots, u_s)$ and $v = (v_1, \ldots, v_p)$. The function

$$L(x, u, v) = f(x) + ug(x) + vh(x)$$

is called the Lagrangian function.

Karush-Kuhn-Tucker (KKT) Point [14, p. 94]: A point $(\bar{x}, \bar{u}, \bar{v})$, where $\bar{x} \in \mathbb{R}^n$, $\bar{u} \in \mathbb{R}^s$, and $\bar{v} \in \mathbb{R}^p$, is called a KKT point if the vector function f, g, and h are differentiable at $\bar{x} = (\bar{x}_1, \ldots, \bar{x}_n)$ and

$$\nabla_x L(\bar{x}, \bar{u}, \bar{v}) = \nabla f(\bar{x}) + \bar{u}\nabla g(\bar{x}) + \bar{v}\nabla h(\bar{x}) = 0, \tag{A.1}$$

$$g(\bar{x}) \le 0, \tag{A.2}$$

$$h(\bar{x}) = 0, \tag{A.3}$$

$$\bar{u} \ge 0, \tag{A.4}$$

$$\bar{u}g(\bar{x}) = 0. \tag{A.5}$$

The vectors $\bar{u} = (\bar{u}_1, \ldots, \bar{u}_s)$ and $\bar{v} = (\bar{v}_1, \ldots, \bar{v}_p)$ are called the Lagrangian multipliers. Sometimes we say that $\bar{u}_i$ ($\bar{v}_i$) is the Lagrangian multiplier with respect to the constraint $g_i \le 0$ ($h_i = 0$). Note that the value of $\bar{u}_i$ ($\bar{v}_i$) might not be unique.

Constraint qualification (CQ) [14, p. 171] for functions g and h play an important role in nonlinear programming. However, since we deal with linear programming in this paper, CQ holds at any minimal point (local minima are global minima in linear programming). Therefore, we won't introduce the definition of CQ here.

First Order Necessary Optimality Condition [14, p. 173]: Suppose that: (i) $\bar{x}$ is a local minimal solution of problem P, (ii) the vector functions f, g, and h are differentiable at $\bar{x}$, and (iii) CQ holds at $\bar{x}$ for g and h. Then there exist $(\bar{u}, \bar{v})$, where $\bar{u} \in \mathbb{R}^s$ and $\bar{v} \in \mathbb{R}^p$, such that $(\bar{x}, \bar{u}, \bar{v})$ is a KKT point.

First Order Sufficient Optimality Condition [14, p. 162]: Suppose P is a convex programming problem. If a KKT point $(\bar{x}, \bar{u}, \bar{v})$ exists, then $\bar{x}$ is a global minimizer.

If the objective is to $maximize$ $f(x)$, we observe that $\max f(x) = -\min\{-f(x)\}$. The corresponding Lagrangian function is $L(x, u, v) = -(-f(x) + u^\tau g(x) + v^\tau h(x)) = f(x) - u^\tau g(x) - v^\tau h(x)$. Since in our paper all the functions f, g and h are linear, the negative functions of them are still linear and therefore, convex.

Lemma A.1. *Let* $(\bar{x}, \bar{u}, \bar{v})$ *be a KKT point of a linear programming problem* P*. Consider problem B:*

$$\begin{array}{llcll} minimize & f(x) + \bar{u}^k g(x)^k & & & \\ subject\ to & g_i(x) & \leq & 0 & for \quad i = k+1, \ldots, s, \\ & h_i(x) & = & 0 & for \quad i = 1, \ldots, p, \\ & x & \in & \mathbb{R}^n, & \end{array}$$

where $g^k = (g_1, \ldots, g_k)^\tau$*,* $\bar{u}^k = (\bar{u}_1, \ldots, \bar{u}_k)$*, and* $k < s$*. Then problems* P *and* B *have the same optimal values and any optimal solution of problem P is an optimal solution of problem B.*

Proof. Consider a KKT point $(\bar{x}, \bar{u}, \bar{v})$ of the problem P. Let us denote by Z_B the objective function of problem B, $Z_B(x) = f(x) + \bar{u}^k g(x)^k$. For $x \in \mathbb{R}^n$, consider

$$Z_L(x) = L(x, \bar{u}, \bar{v}) = f(x) + \bar{u}g(x) + \bar{v}h(x).$$

In view of the First Order Sufficient and Necessary Conditions, any point $\bar{x}$ is optimal for the linear programming problem P if and only if there exist vectors $\bar{u}$, $\bar{v}$ such that $(\bar{x}, \bar{u}, \bar{v})$ is a KKT point of the problem P. Notice that $\bar{x}$ is also a global minimizer of the linear function $L(x, \bar{u}, \bar{v})$ since $\nabla_x L(\bar{x}, \bar{u}, \bar{v}) = 0$. We denote the feasible sets of P and B by S_P and S_B respectively. Therefore,

$$f(\bar{x}) = min_{x \in S_P} f(x) = Z_L(\bar{x}) = min_{x \in \mathbb{R}^n} Z_L(x). \tag{A.6}$$

Notice that $S_P \subseteq S_B$ implies

$$min_{x \in S_P} Z_B(x) \geq min_{x \in S_B} Z_B(x). \tag{A.7}$$

First, let us consider any point $x \in S_P$. For two vectors a and b we write $a \geq b$ (or $a \leq b$) if this inequality holds for all coordinates. Since $\bar{u} \geq 0$, $g(x) \leq 0$, and $h(x) = 0$, we have that $f(x) \geq Z_B(x) \geq Z_L(x)$. Therefore,

$$min_{x \in S_P} f(x) \geq min_{x \in S_P} Z_B(x) \geq min_{x \in \mathbb{R}^n} Z_L(x). \tag{A.8}$$

Second, for any point $x \in S_B$, since $\bar{u} \geq 0$, $g_i(x) \leq 0$, $i = k+1, \ldots, s$, and $h(x) = 0$, we have that $Z_B(x) \geq Z_L(x)$ and therefore,

$$min_{x \in S_B} Z_B(x) \geq min_{x \in \mathbb{R}^n} Z_L(x). \tag{A.9}$$

In view of (A.7)-(A.9), we have

$$min_{x \in S_P} f(x) \geq min_{x \in S_P} Z_B(x) \geq min_{x \in S_B} Z_B(x) \geq min_{x \in \mathbb{R}^n} Z_L(x). \tag{A.10}$$

In view of (A.2)-(A.5), we have that $f(\bar{x}) = Z_B(\bar{x}) = Z_L(\bar{x})$, which along with (A.6) and (A.10) imply that all the inequalities in (A.10) are equalities and therefore, for any optimal solution $\bar{x}$ of problem P,

$$f(\bar{x}) = min_{x \in S_P} f(x) = min_{x \in S_B} Z_B(x) = Z_B(\bar{x}),$$

which completes the proof. ∎

When P is an LP, $(\bar{x}, \bar{u}, \bar{v})$ is a KKT point if and only if $\bar{x}$ is an optimal solution of P and $(\bar{u}, \bar{v})$ is an optimal solution of the dual problem to P [14, p.115, p.127]. Thus, to find $\bar{u}^k$, we need to solve the LP dual to problem P. However, most of contemporary LP solvers use interior point methods and calculate the primal and dual solutions simultaneously. Therefore, we do not formulate the dual LP in this paper.