Berge’s theorem for noncompact image sets

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For an upper semi-continuous set-valued mapping from one topological space to another and for a lower semi-continuous function defined on the product of these spaces, Berge’s theorem states lower semi-continuity of the minimum of this function taken over the image sets. It assumes that the image sets are compact. For Hausdorff topological spaces, this paper extends Berge’s theorem to set-valued mappings with possible noncompact image sets and studies relevant properties of minima.

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1. Introduction

Let \( X \) and \( Y \) be Hausdorff topological spaces, \( u : X \times Y \rightarrow \mathbb{R} = \mathbb{R} \cup \{-\infty, \infty\} \) and \( \Phi : X \rightarrow 2^Y \setminus \{\emptyset\} \). Consider an optimization problem of the form

\[
v(x) := \inf_{y \in \Phi(x)} u(x, y) \quad \text{for each} \ x \in X;
\]

which appears, for instance, in optimal control and game theory. For Hausdorff topological spaces, the well-known Berge’s Theorem (cf. [1, Theorem 2, p. 116]) has the following formulation.

**Berge’s Theorem** (Hu and Papageorgiou [2, Proposition 3.3, p. 83]). If \( u : X \times Y \rightarrow \mathbb{R} \) is a lower semi-continuous function and \( \Phi : X \rightarrow 2^Y \setminus \{\emptyset\} \) is a compact-valued upper semi-continuous set-valued mapping, then the function \( v : X \rightarrow \mathbb{R} \) is lower semi-continuous.

Luque-Vásquez and Hernández-Lerma [3] provide an example of a continuous \( \Phi \) with possible noncompact sets \( \Phi(x) \) and of a lower semi-continuous function \( u(x, y) \) being inf-compact in \( y \), but \( v(x) \) is not lower semi-continuous. In this paper, we extend Berge’s theorem to possibly noncompact sets \( \Phi(x) \), \( x \in X \). Let \( Gr_Z(\Phi) = \{(x, y) \in Z \times Y : y \in \Phi(x)\} \), where \( Z \subseteq X \). For a topological space \( U \), we denote by \( K(U) \) the family of all nonempty compact subsets of \( U \).

For an \( \mathbb{R} \)-valued function \( f \), defined on a nonempty subset \( U \) of a topological space \( U \), consider the level sets

\[
D_f(\lambda; U) = \{y \in U : f(y) \leq \lambda\}, \quad \lambda \in \mathbb{R}.
\]

We recall that a function \( f \) is lower semi-continuous on \( U \) if all the level sets \( D_f(\lambda; U) \) are closed, and a function \( f \) is inf-compact (also sometimes called lower semi-compact) on \( U \) if all these sets are compact.

**Definition 1.1.** A function \( u : X \times Y \rightarrow \mathbb{R} \) is called \( K \)-inf-compact on \( Gr_X(\Phi) \), if for every \( K \in K(X) \) this function is inf-compact on \( Gr_X(\Phi) \).

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The following theorem is the main result of this paper.

**Theorem 1.2.** If the function \( u : X \times Y \to \mathbb{R} \) is \( K \)-inf-compact on \( Gr_X(\Phi) \), then the function \( v : X \to \mathbb{R} \) is lower semi-continuous.

2. Properties of \( K \)-inf-compact functions and Proof of Theorem 1.2

For an upper semi-continuous set-valued mapping \( \Phi : X \to K(Y) \), the set \( Gr_X(\Phi) \) is closed; Berge [1, Theorem 6, p. 112]. Therefore, for such \( \Phi \), if a function \( u(\cdot, \cdot) \) is lower semi-continuous on \( X \times Y \), then it is lower semi-continuous on \( Gr_X(\Phi) \). Thus, Lemma 2.1(i) implies that Theorem 1.2 is a natural generalization of Berge’s Theorem; see also Remark 2.4 and Lemma 2.5.

**Lemma 2.1.** The following statements hold:

(i) If \( u : X \times Y \to \mathbb{R} \) is lower semi-continuous on \( Gr_X(\Phi) \) and \( \Phi : X \to K(Y) \) is upper semi-continuous, then the function \( u(\cdot, \cdot) \) is \( K \)-inf-compact on \( Gr_X(\Phi) \);

(ii) If \( u : X \times Y \to \mathbb{R} \) is inf-compact on \( Gr_X(\Phi) \), then the function \( u(\cdot, \cdot) \) is \( K \)-inf-compact on \( Gr_X(\Phi) \).

**Proof.** Let \( u : X \times Y \to \mathbb{R} \) be lower semi-continuous on \( Gr_X(\Phi) \) and \( \Phi : X \to K(Y) \) be upper semi-continuous. For an arbitrary \( \lambda \in \mathbb{R} \) and an arbitrary fixed \( x \in X \), consider the set

\[ D_{u(x)}(\lambda) = \{ y \in \Phi(x) : u(x, y) \leq \lambda \} \]

\( K \)-inf-compactness of \( u(\cdot, \cdot) \) on \( Gr_X(\Phi) \) implies, that this set is compact.

**Lemma 2.2.** A \( K \)-inf-compact function \( u(\cdot, \cdot) \) on \( Gr_X(\Phi) \) is lower semi-continuous on \( Gr_X(\Phi) \).

**Proof.** Let \( \lambda \in \mathbb{R} \). We need to show that the level set \( D_{u(x)}(\lambda) \) is closed. If this is not true, according to Hu and Papageorgiou [2, Proposition A.1.24(b), p. 893], there exists a net \( (x_n, y_n) \to (x, y) \) in \( X \times Y \) with \( (x_n, y_n) \in D_{u(x)}(\lambda) \) for any \( \alpha \), such that \( (x, y) \not\in D_{u(x)}(\lambda) \). On the other hand, the set \( K = (\bigcup_{\alpha} x_\alpha) \cup [x] \) is compact. Thus, by \( K \)-inf-compactness of \( u(\cdot, \cdot) \) on \( Gr_X(\Phi) \), the level set \( D_{u(x)}(\lambda) \) is compact. Therefore, \( u(\cdot, \cdot) \) is lower semi-continuous on \( Gr_X(\Phi) \).

**Remark 2.4.** Lower semi-continuity of \( u(\cdot, \cdot) \) on \( Gr_X(\Phi) \), inf-compactness of the function \( u(x, \cdot) \) on \( \Phi(x) \) for every \( x \in X \), and continuity of \( \Phi : X \to 2^Y \setminus \{\emptyset\} \) do not imply lower semi-continuity of \( v(\cdot) \); Luque-Vasques and Hernández-Lerma [3].

**Lemma 2.5.** Let \( X \) and \( Y \) be metrizable spaces. Then \( u(\cdot, \cdot) \) is \( \mathbb{K} \)-inf-compact on \( Gr_X(\Phi) \) if and only if the following two conditions hold:

(i) \( u(\cdot, \cdot) \) is lower semi-continuous on \( Gr_X(\Phi) \);

(ii) if a sequence \( \{x_n\}_{n=1,2,...} \) with values in \( \mathbb{X} \) converges and its limit \( x \) belongs to \( \mathbb{X} \) then any sequence \( \{y_n\}_{n=1,2,...} \) with \( y_n \in \Phi(x_n) \), \( n = 1, 2, \ldots \), satisfies the condition that the sequence \( \{u(x_n, y_n)\}_{n=1,2,...} \) is bounded above, has a limit point \( y \in \Phi(x) \).

**Proof.** Let \( u(\cdot, \cdot) \) be \( \mathbb{K} \)-inf-compact on \( Gr_X(\Phi) \). Then, by Lemma 2.3, \( u(\cdot, \cdot) \) is lower semi-continuous on \( Gr_X(\Phi) \). Thus (i) holds. Consider a convergent sequence \( \{x_n\}_{n=1,2,...} \) with values in \( \mathbb{X} \), such that its limit \( x \) belongs to \( \mathbb{X} \). Moreover, let a sequence \( \{y_n\}_{n=1,2,...} \) with \( y_n \in \Phi(x_n) \), \( n = 1, 2, \ldots \), satisfy the condition that the sequence \( \{u(x_n, y_n)\}_{n=1,2,...} \) is bounded above by some \( \lambda \in \mathbb{R} \). Then the set \( K = (\bigcup_{\alpha \geq 1} x_\alpha) \cup [x] \) is compact. Since \( u(\cdot, \cdot) \) is inf-compact on \( Gr_X(\Phi) \), then the sequence \( \{y_n\}_{n=1,2,...} \) belongs to the compact set \( D_{u(x)}(\lambda) \). Therefore this sequence has a limit point \( y \in \Phi(x) \). Thus (ii) holds.

Let (i) and (ii) hold. Fix \( K \in \mathbb{K}(\mathbb{X}) \) and \( \lambda \in \mathbb{R} \). The level set \( D_{u(x)}(\lambda) \) is compact. Indeed let \( \{(x_n, y_n)\}_{n=1,2,...} \subseteq D_{u(x)}(\lambda) \). Since \( K \) is a compact set, the sequence \( \{x_n\}_{n=1,2,...} \) has a subsequence \( \{x_{n_k}\}_{k=1,2,...} \) that converges to its limit point \( x \in \mathbb{X} \). By condition (ii), \( y_{n_k} \in D_{u(x)}(\lambda) \), has a limit point \( y \in \Phi(x) \), that is \( (x, y) \in Gr_X(\Phi) \) is a limit point for the sequence \( \{(x_n, y_n)\}_{n=1,2,...} \subseteq D_{u(x)}(\lambda) \). Lower semi-continuity of \( u(\cdot, \cdot) \) on \( Gr_X(\Phi) \) implies the inequality \( u(x, y) \leq \lambda \). Thus \( (x, y) \in D_{u(x)}(\lambda) \), and the level set \( D_{u(x)}(\lambda) \) is compact. □
Proof of Theorem 1.2. Suppose $v(\cdot)$ is not lower semi-continuous. Then the set $\mathcal{D}_{\text{inf}}(\lambda; \mathcal{X}) = \{ x \in \mathcal{X} : v(x) \leq \lambda \}$ is not closed for some $\lambda \in \mathbb{R}$. According to Hu and Papageorgiou [2, Proposition A.1.24(b), p. 893], there exists a net $x_\alpha \to x$ in $\mathcal{X}$ with $x_\alpha \in \mathcal{D}_{\text{inf}}(\lambda; \mathcal{X})$ for any $\alpha$, such that $x \notin \mathcal{D}_{\text{inf}}(\lambda; \mathcal{X})$. On the other hand, for any $\alpha$ there exists $y_\alpha \in \Phi(x_\alpha)$ such that $v(x_\alpha) = u(x_\alpha, y_\alpha)$. Consider a compact set $K = (\bigcup_{\alpha} \{ x_\alpha \}) \cup \{ x \}$. By virtue of $\mathcal{K}$-inf-compactness of $u(\cdot, \cdot)$ on $\mathcal{G}_\mathcal{X}(\Phi)$, the level set $\mathcal{D}_{\text{inf}}(\lambda; \mathcal{G}_\mathcal{X}(\Phi))$ is compact. Moreover, $\{(x_\alpha, y_\alpha)\} \subseteq \mathcal{D}_{\text{inf}}(\lambda; \mathcal{G}_\mathcal{X}(\Phi))$ for any $\alpha$. Thus, $(x, y) \in \mathcal{D}_{\text{inf}}(\lambda; \mathcal{G}_\mathcal{X}(\Phi))$ for some $y \in \Phi(x)$ and, therefore, $x \in \mathcal{D}_{\text{inf}}(\lambda; \mathcal{X})$. This is a contradiction. □

3. Additional properties of minima

Throughout this section $\mathcal{L}(\mathcal{X})$ denotes the class of all lower semi-continuous functions $\varphi : \mathcal{X} \to \mathbb{R}$ with $\text{dom } \varphi := \{ x \in \mathcal{X} : \varphi(x) \neq -\infty \}$ $\neq \emptyset$. For a topological space $\mathcal{U}$, let $\mathcal{B}(\mathcal{U})$ be the Borel $\sigma$-field on $\mathcal{U}$, that is, the $\sigma$-field generated by all open sets of the space $\mathcal{U}$. For a set $E \subset \mathcal{U}$, we denote by $\mathcal{B}(E)$ the $\sigma$-field whose elements are intersections of $E$ with elements of $\mathcal{B}(\mathcal{U})$. Observe that $E$ is a topological space with induced topology from $\mathcal{U}$, and $\mathcal{B}(E)$ is its Borel $\sigma$-field.

Theorem 3.1. If a function $u(\cdot, \cdot)$ is $\mathcal{K}$-inf-compact on $\mathcal{G}_\mathcal{X}(\Phi)$, then the infimum in (1.1) can be replaced with the minimum, and the nonempty sets $\Phi^+(x), x \in \mathcal{X}$, defined as

$$\Phi^+(x) = \{ a \in \Phi(x) : v(x) = u(x, a) \},$$

satisfy the following properties:

(a) the graph $\mathcal{G}_\mathcal{X}(\Phi^+) = \{ (x, y) : x \in \mathcal{X}, y \in \Phi^+(x) \}$ is a Borel subset of $\mathcal{X} \times \mathcal{Y}$;

(b) if $v(x) = +\infty$, then $\Phi^+(x) = \emptyset$, and, if $v(x) < +\infty$, then $\Phi^+(x)$ is compact.

Proof. $\mathcal{K}$-inf-compactness of $u(\cdot, \cdot)$ on $\mathcal{G}_\mathcal{X}(\Phi)$ implies that infimum in (1.1) can be replaced with the minimum. This follows from Lemmas 2.2 and 2.3, and the classical extreme value theorem [6, Lemma 1.26].

Consider the nonempty sets $\Phi^+(x), x \in \mathcal{X}$, defined in (3.1). The graph $\mathcal{G}_\mathcal{X}(\Phi^+)$ is a Borel subset of $\mathcal{X} \times \mathcal{Y}$, because $\mathcal{G}_\mathcal{X}(\Phi^+) = \{ (x, y) : v(x) = u(x, y) \}$, and the functions $u(\cdot, \cdot)$ and $v(\cdot)$ are lower semi-continuous on $\mathcal{G}_\mathcal{X}(\Phi)$ and $\mathcal{X}$ respectively (Theorem 1.2), and therefore they are Borel.

We remark that, if $v(x) = +\infty$, then $\Phi^+(x) = \emptyset$. If $v(x) < +\infty$, then Lemma 2.2 implies that the set $\Phi^+(x)$ is compact. Indeed, fix any $x \in \mathcal{X}_u := \{ x \in \mathcal{X} : v(x) < +\infty \}$ and set $\lambda = v(x)$. Then the set $\Phi^+(x) = \{ y \in \Phi(x) : u(x, y) \leq \lambda \} = \mathcal{D}_{\text{inf}}(\lambda; \mathcal{X})$ is compact, because $u(\cdot, \cdot)$ is inf-compact on $\Phi(x)$. □

Corollary 3.2 (Cf. Feinberg and Lewis [4, Proposition 3.1]). If a function $u : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ is inf-compact on $\mathcal{G}_\mathcal{X}(\Phi)$, then the function $v : \mathcal{X} \to \mathbb{R}$ is inf-compact and the conclusions of Theorem 3.1 hold.

Proof. The conclusions of Theorem 3.1 directly follow from Lemma 2.1(ii) and Theorems 1.2 and 3.1. The function $v(\cdot)$ is inf-compact, since any level set $\mathcal{D}_{\text{inf}}(\lambda; \mathcal{X})$ is compact as the projection of the compact set $\mathcal{D}_{\text{inf}}(\lambda; \mathcal{G}_\mathcal{X}(\Phi))$ on $\mathcal{X}$, $\lambda \in \mathbb{R}$. □

Let $\mathcal{F} = \{ \phi : \mathcal{X} \to \mathcal{Y} : \phi$ is Borel and $\phi(x) \in \Phi(x)$ for all $x \in \mathcal{X} \}$. A mapping $\phi \in \mathcal{F}$, is called a selector (or a measurable selector).

Theorem 3.3. Let $\mathcal{X}$ and $\mathcal{Y}$ be Borel subsets of Polish (complete separable metric) spaces and $u(\cdot, \cdot)$ be $\mathcal{K}$-inf-compact on $\mathcal{G}_\mathcal{X}(\Phi)$. Then there exists a selector $f \in \mathcal{F}$ such that

$$v(x) = u(x, f(x)), \quad x \in \mathcal{X}.$$ (3.2)

Proof. Let us prove the existence of $f \in \mathcal{F}$ satisfying (3.2). Since the function $v(\cdot)$ is lower semi-continuous (Theorem 1.2), it is Borel and the sets $\mathcal{X}_\infty := \{ x \in \mathcal{X} : v(x) = +\infty \}$ and $\mathcal{X}_v = \mathcal{X} \setminus \mathcal{X}_\infty$ are Borel. Therefore, the $\mathcal{G}_{\mathcal{X}_v}(\Phi^+) = \mathcal{G}_{\mathcal{X}_v}(\Phi) \setminus (\mathcal{X}_\infty \times \mathcal{Y})$. Since the nonempty sets $\Phi^+(x)$ are compact for all $x \in \mathcal{X}_v$, the Arsenin–Kunugui Theorem (cf. [7, p. 297]) implies the existence of a Borel selector $f_v : \mathcal{X}_v \to \mathcal{Y}$ such that $f_v(x) \in \Phi^+(x)$ for all $x \in \mathcal{X}_v$. Consider any Borel mapping $f_2$ from $\mathcal{X}$ to $\mathcal{Y}$ satisfying $f_2(x) \in \Phi(x)$ for all $x \in \mathcal{X}$ and set

$$f(x) = \begin{cases} f_1(x), & \text{if } x \in \mathcal{X}_v, \\ f_2(x), & \text{if } x \in \mathcal{X}_\infty. \end{cases}$$

Then $f \in \mathcal{F}$ and $f(x) \in \Phi^+(x)$ for all $x \in \mathcal{X}$. □

4. Continuity of minima

For a set $U$, denote by $\mathcal{S}(U)$ the family of all nonempty subsets of $U$. A set-valued mapping $F : \mathcal{X} \to \mathcal{S}(\mathcal{Y})$ is upper semi-continuous at $x \in \mathcal{X}$ if, for any neighborhood $\mathcal{G}$ of the set $F(x)$, there is a neighborhood of $x$, say $\mathcal{O}(x)$, such that $F(y) \subseteq \mathcal{G}$ for all $y \in \mathcal{O}(x)$; a set-valued mapping $F : \mathcal{X} \to \mathcal{S}(\mathcal{Y})$ is lower semi-continuous at $x \in \mathcal{X}$ if, for any neighborhood $\mathcal{G}$ of the set $F(x)$, there is a neighborhood of $x$, say $\mathcal{O}(x)$, such that if $y \in \mathcal{O}(x)$, then $F(y) \cap \mathcal{G} \neq \emptyset$; see e.g., Berge [1, p. 109] or Z girovsky et al. [6, Chapter 1, p. 7]. A set-valued mapping is called upper (lower) semi-continuous, if it is upper (lower) semi-continuous at all $x \in \mathcal{X}$. 
Throughout this section we assume that $u(\cdot, \cdot)$ is a real function, that is $u : X \times Y \to \mathbb{R}$. The next theorem provides conditions for a parametric optimization problem to have continuous solutions with regard to the parameter.

**Theorem 4.1.** If $u(\cdot, \cdot)$ is a $K$-inf-compact, continuous function on $G_X(F)$ and $F : X \to S(Y)$ is lower semi-continuous, then the function $v(\cdot)$, defined in (1.1), is continuous on $X$ and the solution multifunction $F^* : X \to K(Y)$ has a closed graph. If, moreover, $F$ is upper semi-continuous, then $F^*$ is upper semi-continuous.

**Proof.** By Theorem 1.2, the function $v(\cdot)$, defined in (1.1), belongs to $L(X)$. Moreover, $v(x) < +\infty$ and, according to Theorem 3.1, $F^*(x) \in K(Y)$ for all $x \in X$. Lower semi-continuity of $F : X \to S(Y)$, upper semi-continuity of $u(\cdot, \cdot)$ on $G_X(F)$, and Hu and Papageorgiou [2, Proposition 3.1, p. 82] imply that $v(\cdot)$ is upper semi-continuous on $X$. Thus, the value function $v(\cdot)$ is continuous on $X$. Since $u(\cdot, \cdot)$ is a continuous function on $G_X(F)$ and $v(\cdot)$ is a continuous function on $X$, the set $G_X(F^*) = \{(x, y) \in G_X(F) : u(x, y) - v(x) \leq 0\}$ is closed. According to Theorem 3.1, $F^*(x) \in K(Y)$ for all $x \in X$.

Now we additionally assume that $F$ is upper semi-continuous. Since $F^*(x) = F^*(x) \cap F(x), x \in X,$ from Berge [1, Theorem 7, p. 112], $F^*$ is upper semi-continuous.

**Theorem 4.1** states that upper semi-continuity of the set-valued mapping $F^*$ is a necessary condition for upper semi-continuity of $v$. According to Luque-Vásques and Hernández-Lerma [3, Theorem 2] (see also [8, Lemma 3.2(f)]), for metric spaces $X$ and $Y$, the function $v(\cdot)$ is lower semi-continuous, if the set-valued mapping $F^* : X \to K(Y)$ is lower semi-continuous, $u(\cdot, \cdot)$ is inf-compact in variable $y$ and lower semi-continuous. The following examples show that lower semi-continuity of the mapping $F^*$ is not necessary for lower semi-continuity of $v(\cdot)$.

**Example 4.2.** The function $v(\cdot)$ is continuous; the real function $u(\cdot, \cdot)$ is $K$-inf-compact on $G_X(F)$ and continuous on $X \times Y$, but it is not inf-compact on $G_X(F)$; the set-valued mapping $F : X \to S(Y)$ is continuous; the set-valued mapping $F^* : X \to K(Y)$ is not lower semi-continuous. Let $X = [0, +\infty), Y = \mathbb{R}, F(x) = (-\infty, x], u(x, y) = \min\{x, y + 1]\}, x \in X, y \in Y$. Then

$$F^*(x) = \begin{cases} \{[-1, 0], & x = 0, \\ \{-1\}, & x > 0, \end{cases} \quad \text{and} \quad v(x) = 0.$$

Therefore, $F^* : X \to K(X)$ is not lower semi-continuous. The function $u(\cdot, \cdot)$ is not inf-compact on $G_X(F)$, since $(x, -1) \in D_{u, \cdot}F(0, G_X(F))$ for each $x \geq 0$.

The following example is similar to Example 4.2, but the function $u$ is inf-compact on $G_X(F)$.

**Example 4.3.** The function $v(\cdot)$ is continuous and inf-compact on $X$; the real function $u(\cdot, \cdot)$ is inf-compact on $G_X(F)$ and continuous on $X \times Y$; the set-valued mapping $F : X \to S(Y)$ is continuous; the set-valued mapping $F^* : X \to K(Y)$ is not lower semi-continuous. Let $X = [0, +\infty), Y = \mathbb{R}, F(x) = (-\infty, x], u(x, y) = \min\{x, y + 1\} + x, x \in X, y \in Y$. Then

$$F^*(x) = \begin{cases} \{[-1, 0], & x = 0, \\ \{-1\}, & x > 0, \end{cases} \quad \text{and} \quad v(x) = x.$$

Therefore, $F^* : X \to K(X)$ is not lower semi-continuous.

The following example shows that continuity properties of $F^*$ may not hold either under the assumptions of Theorem 1.2 or under the stronger assumptions of Berge’s theorem.

**Example 4.4.** The function $v(\cdot)$ is inf-compact on $X$; the real function $u(\cdot, \cdot)$ is inf-compact on $G_X(F)$, but it is not upper semi-continuous on $G_X(F)$; the set-valued mapping $F : X \to S(Y)$ is continuous; the set-valued mapping $F^* : X \to K(Y)$ is neither lower semi-continuous nor upper semi-continuous, and $G_X(F^*)$ is not closed. Let $X = [0, 1], Y = [-1, 1], F(x) = \mathbb{Y}$.

$$u(x, y) = \begin{cases} 0, & x = 0 \text{ and } y \in [-1, 0], \\ y, & x = 0 \text{ and } y \in (0, 1], \\ 1 - y, & x \in (0, 1] \text{ and } y \in [-1, 0], \\ 1, & x \in (0, 1] \text{ and } y \in (0, 1]. \end{cases}$$

Then

$$F^*(x) = \begin{cases} [-1, 0], & x = 0, \\ [0, 1], & x \in (0, 1], \end{cases} \quad \text{and} \quad v(x) = \begin{cases} 0, & x = 0, \\ 1, & x \in (0, 1]. \end{cases}$$

Therefore, $F^* : X \to K(X)$ is neither lower semi-continuous nor upper semi-continuous, and $G_X(F^*)$ is not closed.

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