

ON THE DVORETZKY–WALD–WOLFOWITZ THEOREM ON NONRANDOMIZED STATISTICAL DECISIONS*

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(Translated by the authors)

Abstract. The result by Dvoretzky, Wald, and Wolfowitz on the sufficiency of nonrandomized decision rules for statistical decision problems with nonatomic state distributions holds for arbitrary Borel decision sets and for arbitrary measurable loss functions.

Key words. decision rule, nonatomic measure, nonrandomized decision rule, equivalent decision rules

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Consider the following statistical decision problem studied by Dvoretzky, Wald, and Wolfowitz [5], [6]. Let (X, \mathcal{X}) be a Borel space and let $\Omega = \{\mu_1, \mu_2, \dots, \mu_N\}$ be a finite number of probability measures on (X, \mathcal{X}) . The true probability distribution μ on (X, \mathcal{X}) is not known, but it is known that $\mu \in \Omega$. There is also given a Borel space (A, \mathcal{A}) whose elements a represent the possible decisions. Let $A(x) \in \mathcal{A}$ be the sets of decisions allowable at states $x \in X$. We assume that (a) the graph $Gr(A) = \{(x, a) : x \in X, a \in A(x)\}$ is a measurable subset of $X \times A$, and (b) there exists at least one measurable mapping $\varphi : X \rightarrow A$ with $\varphi(x) \in A(x)$ for all $x \in X$. In particular, (b) implies that $A(x) \neq \emptyset, x \in X$. A decision rule π is a regular transition probability from X to A such that $\pi(A(x)|x) = 1$ for all $x \in X$. A decision rule is called nonrandomized if for each $x \in X$ the measure $\pi(\cdot|x)$ is concentrated at one point. A nonrandomized decision rule π is defined by a measurable mapping $\varphi : X \rightarrow A$ such that $\varphi(x) \in A(x)$ and $\pi(\varphi(x)|x) = 1, x \in X$. We call such a mapping a decision function and denote it by φ .

The loss is defined by a vector-function

$$\rho(\mu, x, a) = (\rho_1(\mu, x, a), \dots, \rho_M(\mu, x, a)),$$

where μ is the true value from Ω and $a \in A(x)$. For each $\mu_n \in \Omega$, the function $\rho_m(\mu_n, x, a)$, $m = 1, \dots, M$, is measurable in (x, a) and takes values in $[-\infty, \infty]$.

For any decision rule π and any $\mu_n \in \Omega$, consider the risk vector

$$\mathcal{R}(\mu_n, \pi) = (\mathcal{R}_1(\mu_n, \pi), \dots, \mathcal{R}_M(\mu_n, \pi)),$$

where

$$\mathcal{R}_m(\mu_n, \pi) = \int_X \int_A \rho_m(\mu_n, x, a) \pi(da | x) \mu_n(dx), \quad m = 1, \dots, M.$$

Here and in what follows, all the integrals of a measurable function f are defined as

$$\int f(y) \nu(dy) = F^+ + F^-,$$

where $+\infty + (-\infty) = -\infty$, $F^\pm = \int f^\pm(y) \nu(dy)$, $f^+ = \max\{f, 0\}$, and $f^- = \min\{f, 0\}$.

The decision rules π_1 and π_2 are called equivalent if

$$\mathcal{R}(\mu_n, \pi_1) = \mathcal{R}(\mu_n, \pi_2) \quad \text{for all } n = 1, \dots, N.$$

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A measure ν defined on X is called nonatomic if $\nu(\{x\}) = 0$ for each $x \in X$.

For a finite set Ω of nonatomic measures, Dvoretzky, Wald, and Wolfowitz [5], [6] proved that, when the decision set A is finite, for any decision rule there exists an equivalent nonrandomized decision rule. By approximating an infinite decision set A by finite sets, they also described sufficient conditions when the equivalence holds for an infinite A . According to the following statement, this equivalence holds without any additional assumptions on the decision set A or loss functions ρ .

THEOREM 1. *If all the measures μ_1, \dots, μ_N are nonatomic, then for each decision rule there exists an equivalent nonrandomized decision rule.*

When ρ is a number rather than a vector ($M = 1$) and the decision set A is finite, Theorem 1 is Theorem 3.1 in [6]; see also [5] and [10, p. 79]. Dvoretzky, Wald, and Wolfowitz [6, section 4] also extended their result to the case when A is compact in the specific metric associated with the function ρ . Balder [2, Corollary 2.5] proved Theorem 1 for the function ρ being continuous in a and uniformly bounded. Balder's result [2, Corollary 2.5] is a generalization of the results in [6, section 4]. Blackwell [4] proved for problems with Borel decision sets that for a mixture of a finite number of nonrandomized decision rules there exists an equivalent nonrandomized decision rule.

We notice that in [5], [6] X was considered as an n -dimensional Euclidean space with the remark that that assumption was not essential. In [2], (X, \mathcal{X}) was an arbitrary measurable space. It appears that the assumption that X is a Borel space covers all realistic applications.

For the readers' convenience, we indicate here the major differences in notation between this paper and [6]. We denote possible state distributions by μ_1, \dots, μ_N while they were denoted by F_1, \dots, F_p in [6]. The decision set A and decisions $a \in A$ correspond to D and $d \in D$ in [6], respectively. We use the notation π for a decision rule instead of δ in [6]. The loss function $\rho(\mu, x, a)$ corresponds to $W(F, d, x)$ and the risk function $\mathcal{R}(\mu, \pi)$ corresponds to $r(F, \delta)$ in [6]. We remark that our model formulation is slightly more general than in [5], [6] because (a) we consider the vector of loss functions ρ , and (b) we allow feasible decision sets $A(x)$ at states $x \in X$ to be different at different states x .

Theorem 1 follows from [9, Theorem 2.1]. First, we present Theorem 2.1 of [9] here. Consider a measurable vector-valued function $r(x, a) = (r^1(x, a), \dots, r^L(x, a))$ with $-\infty \leq r^\ell(x, a) \leq \infty$, $\ell = 1, \dots, L$. For a probability measure $\hat{\mu}$ on X and for a decision rule π , let

$$R(\hat{\mu}, \pi) = (R^1(\hat{\mu}, \pi), \dots, R^L(\hat{\mu}, \pi)),$$

where

$$R^\ell(\hat{\mu}, \pi) = \int_X \int_A r^\ell(x, a) \pi(da | x) \hat{\mu}(dx), \quad \ell = 1, \dots, L.$$

The following statement is [9, Theorem 2.1] for a one-step model; see [3, Theorem 5.3] for a relevant statement when (X, \mathcal{X}) is a general measurable space.

THEOREM 2. *If the probability measure $\hat{\mu}$ is nonatomic, then for any decision rule π there exists a nonrandomized decision rule φ with $R(\hat{\mu}, \varphi) = R(\hat{\mu}, \pi)$.*

Proof of Theorem 1. We define

$$(1) \quad \hat{\mu} = \frac{1}{N} (\mu_1 + \dots + \mu_N),$$

$$(2) \quad r^{n,m}(x, a) = \rho_m(\mu_n, x, a) \frac{d\mu_n}{d\hat{\mu}}(x), \quad n = 1, \dots, N, \quad m = 1, \dots, M.$$

Then

$$\mathcal{R}(\mu_n, \pi) = (R^{n,1}(\hat{\mu}, \pi), \dots, R^{n,M}(\hat{\mu}, \pi)), \quad n = 1, \dots, N,$$

and Theorem 1 follows from Theorem 2 when the initial distribution is $\hat{\mu}$ and each decision rule π is characterized by $L = N \times M$ criteria $R^{n,m}(\hat{\mu})$, $n = 1, \dots, N$, $m = 1, \dots, M$.

Remark. An alternative proof of Theorem 1 can be provided by using Blackwell's [4] result that for a mixture of a finite number of nonrandomized decision rules there exists an equivalent nonrandomized decision rule, Caratheodory's theorem, and a one-step version

of the fact that any randomized strategy (a multiple-step decision rule) can be presented as a mixture of nonrandomized strategies (see [8, Theorem 5.1, Theorem 5.2] or [7, Theorem 1]). Presentation of randomized strategic measures via mixtures of strategic measures corresponding to nonrandomized strategies was introduced by Aumann [1]. We also notice that, in the case $A(x) \equiv A$ considered by Blackwell [4], for a one-step model the fact that any randomized decision rule can be presented as a mixture of nonrandomized decision rules follows from the Krein–Milman theorem. Indeed, in this case, we can set $A = [0, 1]$. Then the set of all measures on $X \times A$ with a given initial state distribution is a compact, the extreme points of which correspond to nonrandomized decision rules; see [11, Theorem 10, p. 83] for details.

The following example demonstrates an application of Theorem 1.

Example. Consider an inventory control problem in which the goal is to match random demand $X = X_1 + X_2$, where X_1 and X_2 are independent and identically distributed random variables with a nonatomic distribution μ . This distribution is unknown, but it is known that $\mu \in \{\mu_1, \mu_2, \dots, \mu_N\}$, where each μ_n is nonatomic. Based on the observation x of X_1 , the decision maker must choose the order size $a \in A = \mathbf{R}^1$. The final loss equals $C(x, y, a)$, where y is the realization of X_2 . For instance, $C(x, y, a) = C_1(a) + C_2(x + y - a)$, where C_1 is the ordering cost and C_2 reflects how well the order matches the demand. The total expected loss (risk) equals

$$\mathcal{R}(\mu_n, \pi) = \int_{\mathbf{R}^1} \int_{\mathbf{R}^1} \rho(\mu_n, x, a) \pi(da | x) \mu_n(dx), \quad \mu_n \in \Omega,$$

where

$$\rho(\mu_n, x, a) = \int_{\mathbf{R}^1} C(x, y, a) \mu_n(dy).$$

Now, according to Theorem 1, any possible vector of risks $(\mathcal{R}(\mu_1, \pi), \dots, \mathcal{R}(\mu_N, \pi))$ can be achieved by a nonrandomized decision rule φ . Note that the results of [2], [5], and [6] are not applicable here if the function C is not continuous.

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