HW3 HW3

2/ Let $E$ be the event that in the blood of the randomly selected soldier A antigen is found. Let $F$ be the event that the blood type of the soldier is A. We have

$$P(F|E) = \frac{P(FE)}{P(E)} = \frac{0.41}{0.41 + 0.04} = 0.911.$$

6/ Both of the inequalities are equivalent to $P(AB) > P(A)P(B)$.

10/ $$P(\alpha = i|\beta = 0) = \begin{cases} \frac{1}{19} & \text{if } i = 0 \\ \frac{2}{19} & \text{if } i = 1, 2, 3, \ldots, 9 \\ 0 & \text{if } i = 10, 11, 12, \ldots, 18. \end{cases}$$

4/ (a) $\frac{8}{20} \times \frac{7}{19} \times \frac{6}{18} \times \frac{5}{17} = 0.0144.$

(b) $\frac{8}{20} \times \frac{7}{19} \times \frac{12}{18} + \frac{8}{20} \times \frac{12}{19} \times \frac{7}{18} + \frac{12}{20} \times \frac{8}{19} \times \frac{7}{18} + \frac{8}{20} \times \frac{7}{19} \times \frac{6}{18} = 0.344.$

7/ Let $A_i$ be the event that the $i$th person draws the 'you lose' paper. Clearly,

$$P(A_1) = \frac{1}{200},$$

$$P(A_2) = P(A_1^c A_2) = P(A_1^c)P(A_2 \mid A_1^c) = \frac{199}{200} \cdot \frac{1}{199} = \frac{1}{200},$$

$$P(A_3) = P(A_1^c A_2^c A_3) = P(A_1^c)P(A_2^c \mid A_1^c)P(A_3 \mid A_1^c A_2^c) = \frac{199}{200} \cdot \frac{198}{199} \cdot \frac{1}{198} = \frac{1}{200},$$

6/ $(0.20)(0.40) + (0.35)(0.60) = 0.290.$

14/ Let $M$, $C$, and $F$ denote the events that the random student is married, is married to a student at the same campus, and is female, respectively. We have that

$$P(F|M) = P(F|MC)P(C|M) + P(F|MC^c)P(C^c|M) = (0.40)\frac{1}{3} + (0.30)\frac{2}{3} = 0.333.$$

18/ Let $E$ be the event that the third number falls between the first two. Let $A$ be the event that the first number is smaller than the second number. We have that

$$P(E|A) = \frac{P(EA)}{P(A)} = \frac{1/6}{1/2} = \frac{1}{3}.$$

Intuitively, the fact that $P(A) = 1/2$ and $P(EA) = 1/6$ should be clear (say, by symmetry). However, we can prove these rigorously. We show that $P(A) = 1/2$; $P(EA) = 1/6$ can be proved similarly. Let $B$ be the event that the second number selected is smaller than the first number. Clearly $A = B^c$ and we only need to show that $P(B) = 1/2$. To do this, let $B_i$ be the event that the first number drawn is $i$, $1 \le i \le n$. Since $\{B_1, B_2, \ldots, B_n\}$ is a partition of the sample space,

$$P(B) = \sum_{i=1}^{n} P(B|B_i)P(B_i).$$

Now $P(B|B_1) = 0$ because if the first number selected is 1, the second number selected cannot be smaller. $P(B|B_i) = \frac{i-1}{n-1}$, $1 \le i \le n$ since if the first number is $i$, the second number must be one of $1, 2, 3, \ldots, i-1$ if it is to be smaller. Thus

$$P(B) = \sum_{i=1}^{n} P(B|B_i)P(B_i) = \sum_{i=2}^{n} \frac{i-1}{n-1} \cdot \frac{1}{n} = \frac{1}{(n-1)n} \sum_{i=2}^{n} (i-1)$$

$$= \frac{1}{(n-1)n}\big[1 + 2 + 3 + \cdots + (n-1)\big] = \frac{1}{(n-1)n} \cdot \frac{(n-1)n}{2} = \frac{1}{2}.$$

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2/ $\dfrac{1(2/3)}{1(2/3)+(1/4)(1/3)}=\dfrac{8}{9}$.

14/ Let $I$ be the event that the person is ill with the disease, $N$ be the event that the result of the test on the person is negative, and $R$ denote the event that the person has the rash. We are interested in $P(I\mid R)$:

$$P(I\mid R)=P(IN\mid R)+P(IN^c\mid R)=O+P(IN^c\mid R).$$

Since $\{IN, IN^c, I^cN, I^cN^c\}$ is a partition of the sample space, by Bayes' Formula,

$$\begin{aligned} P(I\mid R)&=P(IN^c\mid R)\\ &=\frac{P(R\mid IN^c)P(IN^c)}{P(R\mid IN)P(IN)+P(R\mid IN^c)P(IN^c)+P(R\mid I^cN)P(I^cN)+P(R\mid I^cN^c)P(I^cN^c)}\\ &=\frac{(0.2)(0.30\times 0.90)}{0(0.30\times 0.10)+(0.2)(0.30\times 0.90)+0(0.70\times 0.75)+(0.2)(0.70\times 0.25)}=0.61. \end{aligned}$$

8/ For $1\le i\le 4$, let $A_i$ be the event of obtaining 6 on the $i$th toss. Chevalier de Méré had implicitly thought that $A_i$'s are mutually exclusive and so

$$P(A_1\cup A_2\cup A_3\cup A_4)=\frac{1}{6}+\frac{1}{6}+\frac{1}{6}+\frac{1}{6}=4\times\frac{1}{6}.$$

Clearly $A_i$'s are not mutually exclusive. The correct answers are $1-(5/6)^4=0.5177$ and $1-(35/36)^{24}=0.4914$.

19/ (a) By Theorem 1.6,

$$\begin{aligned} P\big(A(B\cup C)\big)&=P(AB\cup AC)=P(AB)+P(AC)-P(ABC)\\ &=P(A)P(B)+P(A)P(C)-P(A)P(B)P(C)\\ &=P(A)\big[P(B)+P(C)-P(B)P(C)\big]=P(A)P(B\cup C). \end{aligned}$$

(b) $P\big((A-B)C\big)=P(AB^cC)=P(A)P(B^c)P(C)=P(AB^c)P(C)=P(A-B)P(C)$.

38/ Let $E_i$ be the event that the switch located at $i$ is closed. We want to calculate the probability of $E_2E_4\cup E_1E_5\cup E_2E_3E_5\cup E_1E_3E_4$. Using the rule to calculate the probability of the union of several events (the inclusion-exclusion principle) we get that the answer is $2p^2+2p^3-5p^4+p^5$.