AMS-511 Foundations of Quantitative Finance

Fall 2018 — Mid-Term Solutions

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Question 1 (20 pts.)

John borrows \$10,000 from Mary and agrees to repay her in two years with interest calculated at 7%. At the end of the two-year period John comes to Mary and says that he can only pay \$8,000 towards the loan and asks that she accept the remaining balance, with interest, one year later. They come to an agreement that the interest rate will be 8% for the remaining period. Assume continuous compounding in all calculations.

- What is the balance at the end of two-years, not including the \$8,000 payment?
- Assuming the \$8,000 payment is made at the end of two-years, what is the remaining balance at the end of the third year?

Mary wants to compute the overall interest rate (*i.e.*, her internal rate of return or IRR) for the two payments received over the three-year period. Assume continuous compounding in all calculations.

- Draw a cash-flow diagram representing the cash-flows of this transaction. You can do this from either John's or Mary's point of view.
- What is the equation she must set up to solve for the IRR for these cash-flows? You can either represent it using standard mathematical notation or as *Mathematica* code. It is not necessary to actually solve for the IRR.

Solution

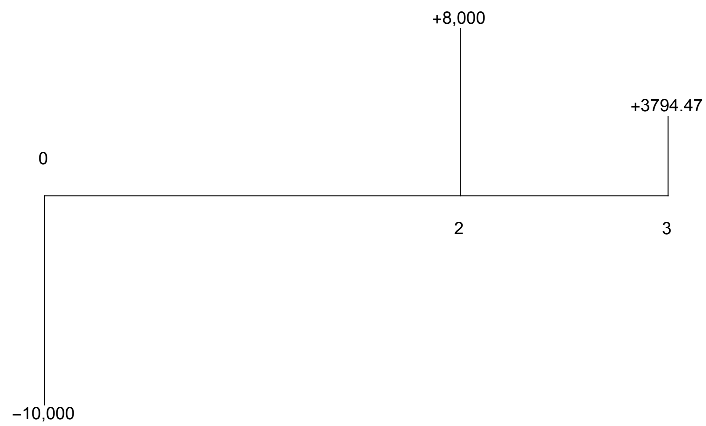
- The balance at the end of two years B_2 is:

$$B_2 = 10\,000\, e^{2\times 0.07} = 11\,502.70$$

- The balance at the end of the third year B_3 is the FV of remaining balance of B_2 after the \$8,000 payment:

$$B_3 = (B_2 - 8000)\, e^{0.08} = 3794.47$$

- The cash-flow diagram from Mary's (the lender's) perspective is:



From John's (the borrower's) perspective, the amounts would be reversed. Either is valid.

- The IRR is that common return which applied to the cash-flows equals 0. We could express the IRR as the solution for y in the non-linear equation:

$$0 = -10\,000 + 8000\, e^{-2\,y} + 3794.47\, e^{-3\,y}$$

Any of a number of equivalent formulations would work. For example using FindRoot[ ], we could formulate this as:

```
In[1]:= FindRoot[0 == -10 000. + 8000. e^(-2. y) + 3794.47 e^(-3. y), {y, 0.075}]

Out[1]= {y → 0.0713248}
```

where the solution (not required on the test) is 7.13%. Or the form of the equation could be:

$$10\,000 = 8000\, e^{-2\,y} + 3794.47\, e^{-3\,y}$$

Question 2. (20 pts.)

You are given the following spot rate curve data:

Years	Spot
1	0.020
2	0.027
3	0.032
4	0.034

Assume annual compounding and base your answers on the given spot rates:

- What is the yield on a $1,000 three-year zero coupon bond; *i.e.*, a bond which pays a single payment of $1,000 at the end of three years?
- What is the price of the bond above at origination?
- What is the discount rate for a cash payment made four years forward in time?
- What is the price of a two-year bond with a face value of $100,000 and annual coupon of $2,500?
- Compute the forward rate $f_{1,2}$.

Solution

- The yield on a three-year zero coupon bond is simply the spot rate, 3.2%:

$$S = \frac{1000}{(1+s_3)^3} = \frac{1000}{(1+y)^3}$$

- The price of the bond above at origination is the PV of its face amount:

$$S(0) = \frac{c_3}{(1+s_3)^3} = \frac{1000}{(1+0.032)^3} = 909.83$$

- The discount factor d_4 is:

$$d_4 = \frac{1}{(1+s_4)^4} = 0.8748$$

- The price of a two-year bond with a face value of \$100,000 and annual coupon of \$2,500 is

$$S = \frac{c}{1+s_1} + \frac{c+F}{(1+s_2)^2} = \frac{2500}{1+0.020} + \frac{2500+100\,000}{(1+0.027)^2} = 99\,632.30$$

- The forward rate $f_{1,2}$ is:

$$f_{i,j} = \left(\frac{(1+s_j)^j}{(1+s_i)^i}\right)^{1/(j-i)} - 1 = \left(\frac{(1+0.027)^2}{1+0.020}\right)^{1/(2-1)} - 1 = 0.0340$$

Question 3. (10 pts.)

A bond with a face value of \$10,000, a semi-annual coupon of \$250 and exactly 3 years to maturity has a market yield quoted at 5.2%. Assuming that there is no accrued interest. What is the price of the bond?

Solution

$$\begin{aligned} S &= \frac{F}{(1+y/k)^{k\,n}} + \frac{c}{y/k}\left(1 - \frac{1}{(1+y/k)^{k\,n}}\right) \\ &= \frac{10\,000}{(1+0.052/2)^{2\times 3}} + \frac{250}{0.052/2}\left(1 - \frac{1}{(1+0.052/2)^{2\times 3}}\right) \\ &= 9945.10 \end{aligned}$$

Question 4. (10 pts.)

Two uncorrelated risk assets have mean returns of 8% and 10%, respectively, and standard deviations of return of 10% and 12%, respectively. The two risk assets are uncorrelated. The risk-free rate of return is 5%. Assume that you have one unit of capital to allocate and that there is no short selling.

- What allocation of the risk assets represents the minimum variance portfolio?
- What is the mean and standard deviation of the minimum variance portfolio?

Solution

Note throughout that the fact the assets are uncorrelated results in considerable computational simplifications.

- The general solution is:

$$\vec{\mathbf{x}} = \lambda\, \boldsymbol{\Sigma}^{-1} \vec{\boldsymbol{\mu}} + \left(\frac{1 - \lambda\, \mathbf{1}^T \boldsymbol{\Sigma}^{-1} \vec{\boldsymbol{\mu}}}{\mathbf{1}^T \boldsymbol{\Sigma}^{-1} \mathbf{1}} \right) \boldsymbol{\Sigma}^{-1} \vec{\mathbf{1}}$$

The minimum variance portfolio has $\lambda = 0$, or

$$\vec{\mathbf{x}} = \frac{\boldsymbol{\Sigma}^{-1} \vec{\mathbf{1}}}{\vec{\mathbf{1}}^T \boldsymbol{\Sigma}^{-1} \vec{\mathbf{1}}}$$

Because the returns are uncorrelated, the covariance matrix $\boldsymbol{\Sigma}$ is a diagonal matrix; hence, the above simplifies to:

$$x_i = \sigma_i^{-2} \Big/ \sum_{j=1}^{n} \sigma_j^{-2}$$

$$\vec{\mathbf{x}} = \begin{pmatrix} \sigma_1^{-2} \\ \sigma_2^{-2} \end{pmatrix} \Big/ \left(\sigma_1^{-2} + \sigma_2^{-2}\right) = \begin{pmatrix} 0.10^{-2} \\ 0.12^{-2} \end{pmatrix} \Big/ \left(0.10^{-2} + 0.12^{-2}\right) = \begin{pmatrix} 0.590 \\ 0.409 \end{pmatrix}$$

- What is the mean and standard deviation of the minimum variance portfolio?

The mean is

$$\mu_P = \vec{\boldsymbol{\mu}}^T \vec{\mathbf{x}} = \begin{pmatrix} 0.08 \\ 0.10 \end{pmatrix}^T \begin{pmatrix} 0.590 \\ 0.409 \end{pmatrix} = 0.0882$$

and again relying on the fact that covariance is a diagonal matrix:

$$\sigma_P = \sqrt{x_1^2 \sigma_1^2 + x_2^2 \sigma_2^2} = \sqrt{0.590^2\, 0.10^2 + 0.409^2\, 0.12^2} = 0.0767$$

Question 5. (30 pts.)

Assume that the CAPM holds for each asset i in market M; *i.e.*, returns follow:

$$r_i(t) - r_f = \beta_i\left(r_M(t) - r_f\right) + \epsilon_i(t)$$

You are given the following annual data on four investments, $i \in \{1, 2, 3, 4\}$ in a market M:

- risk-free: $r_f = 0.02$
- market mean $\mu_M = 0.08$
- market standard deviation $\sigma_M = 0.10$
- covariances to market $\vec{\boldsymbol{\sigma}}_{i,M} = \{0.010, 0.009, 0.006, 0.014\}$
- error standard deviations $\vec{\boldsymbol{\sigma}}_{\epsilon_i} = \{0.08, 0.10, 0.07, 0.09\}$

Under the assumption that the CAPM applies:

- Compute the betas, $\vec{\boldsymbol{\beta}}$, of the assets.

- What is the correlation between asset 2 and the market M?
- Let $\vec{\mathbf{x}}$ represent a portfolio allocation; compute the mean-variance efficient portfolio P such that $\vec{\mathbf{1}}^T \vec{\mathbf{x}} = 1$.
- What is the beta β_P of the portfolio P?
- Compute the mean of portfolio P.
- The investor wishes to keep 20% of its assets in cash and place the remainder in the optimal portfolio P. Assuming returns are Normally distributed what is the mean return for the combined cash-risky portfolio?

Solution

Factor models are not only parsimonious but yield considerable computational advantages. The CAPM can be viewed as a single-factor case.

Note that the fact that we have included the risk-free asset in the CAPM, the optimal portfolio will be the market (*i.e.*, tangent) portfolio when the risky assets are normalized to sum to 1.

- The ordinary least squares relationship of the to-market covariances to the market variance and betas is:

$$\beta_i = \frac{\sigma_{i,M}}{\sigma_M{}^2}$$

$$\vec{\boldsymbol{\beta}} = \frac{\{0.010, 0.009, 0.006, 0.014\}}{0.10^2} = \{1.0, 0.9, 0.6, 1.4\}$$

- The correlation is a function of the covariance and standard deviations:

$$\rho_{i,M} = \frac{\sigma_{i,M}}{\sigma_i\,\sigma_M}$$

We can compute σ_i using the CAPM. Given that the systematic and non-systematic variances are uncorrelated, an asset's standard deviation is:

$$\sigma_i = \sqrt{\beta_i{}^2\,\sigma_M{}^2 + \sigma_{\epsilon_i}{}^2}$$

Thus,

$$\rho_{2,M} = \frac{\sigma_{2,M}}{\left(\sqrt{\beta_2{}^2\,\sigma_M{}^2 + \sigma_{\epsilon_2}{}^2}\right)\sigma_M} = \frac{0.009}{\left(\sqrt{0.9^2 \times 0.10^2 + 0.10^2}\right)\times 0.10} = 0.6690$$

The above can also be derived from the covariance matrix of a factor model, $\mathbf{B}\,\mathbf{B}^T + \mathbf{D}$, treating the CAPM as a single-factor model.

- Under the CAPM the optimal portfolio allocation is proportional to:

$$x_i \propto \frac{\beta_i}{\sigma_{\epsilon_i}{}^2}$$

Normalizing so that the sum is 1 yields:

$$\vec{\mathbf{x}} = \begin{pmatrix} 0.2885 \\ 0.1662 \\ 0.2261 \\ 0.3192 \end{pmatrix}$$

- The portfolio beta is the allocation-weighted asset betas.

$$\beta_P = \vec{\beta}^T \vec{\mathbf{x}} = \begin{pmatrix} 1.0 \\ 0.9 \\ 0.6 \\ 1.4 \end{pmatrix}^T \begin{pmatrix} 0.2885 \\ 0.1662 \\ 0.2261 \\ 0.3192 \end{pmatrix} = 1.0206$$

- The mean of portfolio P can be derived directly from the CAPM and β_P:

$$\mu_P = r_f + \beta_P(\mu_M - r_f) = 0.02 + 1.0206\,(0.08 - 0.02) = 0.0812$$

- The effect of holding 20% in cash and the remainder in the risky portfolio (Call this combined portfolio L) is:

$$\mu_L = \lambda\, r_f + (1 - \lambda)\, \mu_P = 0.20 \times 0.02 + (1 - 0.20)\, 0.0812 = 0.0690$$

Question 6. (10 pts.)

Consider the following questions given the *Mathematica* statements:

- Given the numeric vector $\vec{\mathbf{x}} = \{x_1, x_2, x_3\}$, what is the value of the expression $e^{\vec{\mathbf{x}}} / \vec{\mathbf{x}}$?

  The vector of $\{\ldots, e^{x_i}/x_i, \ldots\}$.

In[2]:= $\dfrac{e^{-\{x1,x2,x3\}}}{\{x1, x2, x3\}}$

Out[2]= $\left\{\dfrac{e^{-x1}}{x1}, \dfrac{e^{-x2}}{x2}, \dfrac{e^{-x3}}{x3}\right\}$

- What does Rest[FoldList[Times, 1, Range[4]]] produce?

  The cumulative product of the sequence of integers from 1 to 4.

```
In[3]:= Rest[FoldList[Times, 1, Range[4]]]
Out[3]= {1, 2, 6, 24}
```

- Describe in words the result of Table[{i, i²}, {i, 1, 3, 0.5}].

  This expression results in a vector of the pair $\{i, i^2\}$ where i varies from 1 to 3 in increments of 0.5.

```
In[4]:= Table[{i, i^2}, {i, 1, 3, 0.5}]
Out[4]= {{1., 1.}, {1.5, 2.25}, {2., 4.}, {2.5, 6.25}, {3., 9.}}
```

- Given the numeric vector $\vec{\mathbf{x}} = \{x_1, x_2, x_3, x_4\}$, what is the result of Rest[#] – Most[#] &[x]?

  This expression produces the first-order differences $\{\ldots, x_i - x_{i-1}, \ldots\}$.

```
In[5]:= Rest[#] - Most[#] &[{x1, x2, x3, x4}]
Out[5]= {-x1 + x2, -x2 + x3, -x3 + x4}
```

- What is the result Total[3 × {2, 4, 6} – 1]?

This expression takes each element of the vector {2, 4, 6} and multiplies it by 3, subtracts 1 and then totals them. The result is 33

In[6]:=
```
Total[3 {2, 4, 6} - 1]
```
Out[6]=
```
33
```

Produce the *Mathematica* code which will:

- Be a function which squares the elements of a numeric vector **x** and then normalizes them to sum to one.

  An inefficient, but acceptable, implementation is:

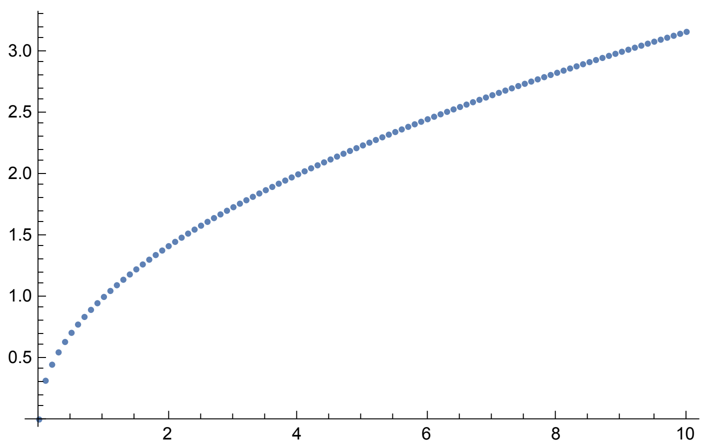
In[7]:=
```
#^2 / Total[#^2] &[{1, 2, 3}]
```
Out[7]= $\left\{\frac{1}{14}, \frac{2}{7}, \frac{9}{14}\right\}$

We can, however, avoid taking the square twice by nesting two functions”

In[8]:=
```
# / Total[#] &[#^2] &[{1, 2, 3}]
```
Out[8]= $\left\{\frac{1}{14}, \frac{2}{7}, \frac{9}{14}\right\}$

- Produce a plot of x versus y where $y = \sqrt{x}$ and x varies from 0 to 10 in increments of 0.1.

In[9]:=
```
ListPlot[Table[{x, Sqrt[x]}, {x, 0, 10, 0.1}]]
```
Out[9]=



- Given two vectors **u** and **v**, both of length n, computes the vector **z** with components

$$z_i = \log\,(u_i - v_i)^2$$

where, for example, the scalar u_i represents the i^{th} element of the vector **u**.

In[10]:=
```
u = {u1, u2, u3};
v = {v1, v2, v3};
Log[(u - v)]^2
```
Out[12]=
```
{Log[u1 - v1]^2, Log[u2 - v2]^2, Log[u3 - v3]^2}
```

- Solve the non-linear equation $\text{Cos}[\vartheta] == \vartheta^2$.

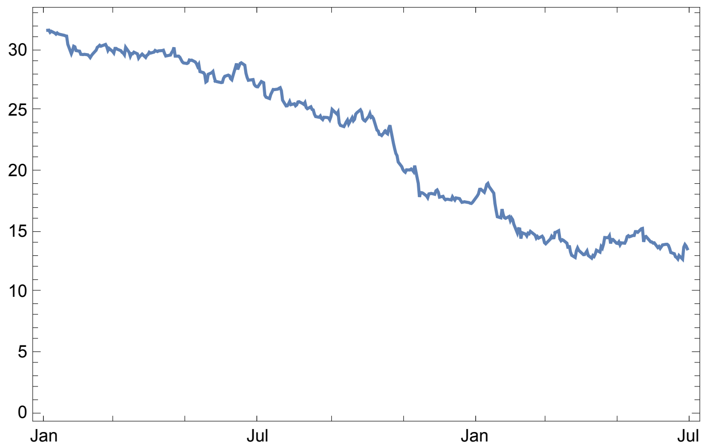
In[13]:=
```
FindRoot[Cos[ϑ] == ϑ^2, {ϑ, 0.5}]
```
Out[13]=
```
{ϑ → 0.824132}
```

- Download the closing prices of General Electric (ticker symbol “GE”) for the period from January 1, 2017 to June 30, 2018.

Although it was not part of the exam, it is instructive to plot the price of GE over the selected period.

In[14]:=
```
mnGE = FinancialData["GE", {{2017, 1, 1}, {2018, 6, 30}}];
DateListPlot[mnGE]
```

Out[15]=



The drop in GE prices over the period may explain why its CEO has been recently replaced.