AMS 511 - Foundations

Class 5 - Utility and Pricing Theory

Robert J. Frey
Research Professor
Stony Brook University, Applied Mathematics and Statistics
frey@ams.sunysb.edu
This material introduces the concepts of utility functions and risk neutral pricing. These are the two main theories used in evaluating uncertain future cash flows.
February, 2006

1 - Utility Functions

1.1 - Utility Functions

A utility function is a function $U : \mathcal{R}_1 \rightarrow \mathcal{R}_1$, generally interpreted as a mapping from wealth to utility. The utility function provides a means for ranking the desirability of all possible wealth values.

- Risk Neutral: $U(x) = x$

![Risk Neutral Graph]

- Exponential: $U(x) = -e^{-bx}$, for $b > 0$

![Exponential Graph]
• Logarithmic: \( U(x) = \ln x, \) for \( x > 0 \)

![Logarithmic Graph](image1)

• Power: \( U(x) = b x^b, \) for \( b \leq 1 \) and \( b \neq 0 \)

![Power Graph](image2)

• Quadratic: \( U(x) = x - b x^2, \) for \( b > 0 \)

![Quadratic Graph](image3)

Example - Anyone who buys a $1 lottery ticket for a chance to win a million dollars knows that he or she is not likely to win. But the loss of a dollar has little impact on the person's lifestyle while winning the million dollars would have a profound effect. A person may quite rationally decide to play the lottery if their utility function takes such perceptions into account.

Example - A retiree may understand that the expected rate of return on the stock market is higher than that of a portfolio of government bonds. But the bonds will (almost certainly) not default and will (almost certainly) deliver their coupons on schedule while the stock market may go down dramatically causing losses that will not be recouped for years—perhaps not within the lifetime of the retiree. That retiree may then quite rationally decide that the negative utility of the possible lost income from stocks outweighs the possible upside and decides to invest her funds in government bonds despite the lower expected rate of return.
1.2 - Equivalence Under Affine Transformation

An affine transformation of a utility function, \( V(x) = \alpha + \beta U(x) \), gives an equivalent utility function. Note that \( \mathbb{E}[V(x)] = \alpha + \beta \mathbb{E}[U(x)] \), so \( \mathbb{E}[U(X)] \geq \mathbb{E}[U(Y)] \iff \mathbb{E}[V(X)] \geq \mathbb{E}[V(Y)] \). It is in this sense, that the rankings given by \( U \) and \( V \) are identical, that the two functions are considered equivalent.

1.3 - Risk Aversion

Concave Utility and Risk Aversion

A utility function \( U(x) \) defined on an interval \([a, b]\) is said to be \textit{concave} on that interval if \( \forall \ x, y \in [a, b] \) and \( 0 \leq \alpha \leq 1 \)

\[
U(\alpha x + (1 - \alpha) y) \geq \alpha U(x) + (1 - \alpha) U(y)
\]  
(1)

If a utility function is concave on an interval it is said to be \textit{risk averse} on the interval. If concave everywhere it is said to be \textit{risk averse}. If \textit{strictly} concave, i.e., if

\[
U(\alpha x + (1 - \alpha) y) > \alpha U(x) + (1 - \alpha) U(y)
\]  
(2)

then it is said to be \textit{strictly risk averse}.

Arrow-Pratt Absolute Risk Aversion Coefficient

The more concave the utility function the more risk averse. This can be measured by the \textit{Arrow-Pratt absolute risk aversion coefficient}.

\[
a(x) = -\frac{U''(x)}{U'(x)}
\]  
(3)

- Risk Neutral: \( U(x) = x \implies a(x) = 0 \)
- Exponential: \( U(x) = -e^{-bx} \), for \( b > 0 \implies a(x) = b \)
- Logarithmic: \( U(x) = \ln x \), for \( x > 0 \implies a(x) = x^{-1} \)
- Power: \( U(x) = bx^b \), for \( b \leq 1 \) and \( b \neq 0 \implies a(x) = (1 - a) / x \)
- Quadratic: \( U(x) = x - bx^2 \), for \( b > 0 \implies a(x) = 2a / (1 - 2a) \)

Certainty Equivalence

The certainty equivalent is the amount of a certain (risk free) wealth that has the same utility as the expected utility of the wealth \( x \).

\[
U(C) = \mathbb{E}[U(X)]
\]  
(4)

Intuitively, this means we would be “just as happy” with the certain outcome \( C \) as we would with the random outcome \( X \). For a normally risk adverse investor

\[
C \leq E[X]
\]  
(5)
1.4 - Calibration of Utility Functions

Certainty Equivalent of a Lottery

Consider a lottery in which the payoff $X = A$ with probability $p$ and $X = B$ with probability $(1 - p)$. An individual is asked for the certainty equivalent value for the lottery for various values of $p$.

Typically, the following relationship holds.

$$C \leq E[X] = pA + (1 - p)B$$  \hspace{1cm} (6)

We’ll describe the procedure by given a specific example. Consider the case in which $A = 1$ and $B = 2$. The following table of data is collected from the decision maker. The expected payoffs versus certainty equivalents are also plotted.

<table>
<thead>
<tr>
<th>$p$</th>
<th>$E[X]$</th>
<th>$C$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>0.1</td>
<td>1.1</td>
<td>1.00561</td>
</tr>
<tr>
<td>0.2</td>
<td>1.2</td>
<td>1.02473</td>
</tr>
<tr>
<td>0.3</td>
<td>1.3</td>
<td>1.06056</td>
</tr>
<tr>
<td>0.4</td>
<td>1.4</td>
<td>1.11603</td>
</tr>
<tr>
<td>0.5</td>
<td>1.5</td>
<td>1.19381</td>
</tr>
<tr>
<td>0.6</td>
<td>1.6</td>
<td>1.29642</td>
</tr>
<tr>
<td>0.7</td>
<td>1.7</td>
<td>1.42619</td>
</tr>
<tr>
<td>0.8</td>
<td>1.8</td>
<td>1.58534</td>
</tr>
<tr>
<td>0.9</td>
<td>1.9</td>
<td>1.77594</td>
</tr>
<tr>
<td>1.0</td>
<td>2.0</td>
<td>2.0</td>
</tr>
</tbody>
</table>

![Diagram showing the relationship between $C$, $E[X]$, and $p$.]
We have two degrees of freedom in the utility so we can set \( U(A) = A \) and \( U(B) = B \). This means that \( pA + (1 - p)B = pU(A) + (1 - p)U(B) \) or equivalently \( E[X] = E[U(X)] \). Recall that, by definition, \( U(C) = E[U(X)] \). Finally, we can assert that \( U(C) = E[X] \) and, in turn, \( C = U^{-1}(E[X]) \). Thus, the inverse of the function represented by \( C \) is the utility function.

![Graph](ams-511-lec-05-p.nb)

**Questionnaire Method**

An individual’s risk tolerance is a matter of personal choice. Often professional investment managers attempt to gain insight into an individual’s implied utility function by asking a series of questions which gauge the individuals reactions under various scenarios.

**Normative Arguments**

If you lose 50%, then to get back to where you started you have to make 100%. In general, a gain of \( u \) is offset by a loss of \( 1/u \). Thus, it appears reasonable to assert that \( U(u) = -U(1/u) \). The logarithmic utility function \( U(x) = \ln x \) satisfies this requirement. This is a normative argument in that it attempts to determine what a rational utility should be.

**Parameterizing by Curve Fitting**

Once data are collected, then it often makes sense to summarize the utility function with as few parameters as possible. For example, in the “lottery” example above we could fit an exponential utility function using three parameters.

\[
V(x) = a_0 - a_1 e^{-a_2 x} = 2.0 - 16.6 e^{-2.9 x}
\]  

(7)

![Graph](ams-511-lec-05-p.nb)

An affine transformation yields an equivalent result, and we could then simplify our function to a single parameter.

\[
U(x) = -e^{-2.9 x}
\]  

(8)
1.5 - The Mean-Variance Criterion and Utility

**Under Quadratic Utility**

Define the quadratic utility function in the form

\[ U(x) = ax - \frac{1}{2} b x^2, \quad a > 0 \text{ and } b \geq 0 \]  

(9)

then the expected utility of a portfolio whose payoff is the random variable \( W \) is

\[ E[U(W)] = a E[W] - \frac{1}{2} b E[W]^2 - \frac{1}{2} b \text{Var}[W] \]  

(10)

The our goal is to select a feasible portfolio such that \( E[U(W)] \) is maximized. If \( \mu_p \) is the expected portfolio return and \( \sigma_p^2 \) its variance and we assume WOLG that the intial wealth is 1, then

\[ E[W] = 1 + \mu_p \text{ and } \text{Var}[W] = \sigma_p^2 \]  

(11)

For \( x \leq a / b \), \( U(x) \) is a non-decreasing function of \( x \). Under that condition, maximizing \( a E[W] - \frac{1}{2} b E[W]^2 \) is equivalent to maximizing \( E[\mu_p] \) and maximizing \( -\frac{1}{2} b \text{Var}[W] \) is equivalent to minimizing \( \sigma_p^2 \). Thus, maximizing \( E[U(W)] \) selects an efficient portfolio.

**Normal Distributed Returns and Risk Averse Utility Functions**

If asset returns are Normally distributed then the return distribution of a portfolio of such assets can be completely characterized in terms of its mean and variance. Utility functions representing “normal” risk averse behavior are concave, and given two portfolios with the same mean, a risk averse investor will *always* prefer the one with the smaller variance. Thus, under these general assumptions, mean-variance efficient portfolios maximize expected utility.
2 - Pricing Theory

2.1 - Arbitrage and Linear Pricing

Type A Arbitrage and Linear Pricing

_Type A arbitrage_ is said to exist if an investment pays an immediate payoff with no future payoff. Given two securities with prices \( p_1 \) and \( p_2 \). The price of a portfolio of the two securities must be \( p_1 + p_2 \). If it were not then it would be possible, then it would be possible to buy (sell) the portfolio, break it up into its constituent assets and sell (buy) then individually, realizing an immediate riskless profit. It is hard to imagine a liquid and transparent market allowing such a situation to exist because the prices would rapidly (theoretically, instantaneously) adjust to make such a payoff impossible.

Type B Arbitrage

A related form of arbitrage is _type B arbitrage_ in which an investor pays nothing or a negative amount up front and has the prospect of receiving something in the future. This is equivalent to receiving a free lottery ticket.

2.2 - Utility Theory and Portfolio Choice

\[
\begin{align*}
    u^* &= \max_x \left\{ E[U(w^T x)] \mid x \geq 0, \ p^T x = W \right\} \\
    \mathcal{L} &= E[U(w^T x)] - \lambda(p^T x - W)
\end{align*}
\]  

(12) \hspace{1cm} (13)

Differentiating with respect to \( x \) gives for the optimal portfolio \( x^* \)

\[
E[U'(w^T x^*) w_i] = \lambda p_i, \quad \text{for } i = 1, 2, \ldots, n
\]

(14)

If there exists a risk free asset with total return \( 1 + r_f \), then this asset also satisfies (14); hence,

\[
\lambda = E[U'(w^T x^*)](1 + r_f)
\]

(15)

Substituting into (14) yields

\[
p_i = \frac{1}{(1 + r_f)} \frac{E[U'(w^T x^*) w_i]}{E[U'(w^T x^*)]}
\]

(16)

This is the portfolio pricing equation.

2.3 - Log Optimal Pricing

Set \( W = 1 \) and \( U(x) = \ln x \). Since the first derivative of \( \ln x \) is \( 1/x \), we can restate (14) as

\[
E\left[ \frac{w_i}{w^T x^*} \right] = \lambda p_i
\]

(17)

but \( w^T x^* \) is just the total return on the log optimal portfolio which we will represent by \( 1 + r^* \)

\[
E\left[ \frac{w_i}{1 + r^*} \right] = \lambda p_i
\]

(18)

but (18) is valid for the log optimal portfolio itself; hence, \( \lambda = 1 \) and
\[ E\left[ \frac{w_i}{1 + r^u} \right] = p_i \]  

(19)

2.4 - Finite State Models

The text covers this in more generality; however, we’ll work through a simple binomial case here.

In the absence of arbitrage and in the presence of a riskfree asset we have

\[ \begin{pmatrix} 1 & 1 + r_f \ 1 + r_f \ p^+ [1] & p^- [1] \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} > \begin{pmatrix} 0 \\ 0 \end{pmatrix} \]  

(20)

Note that \( p^+ [1]/p[0] > 1 + r_f > p^- [1]/p[0] \) implies that \( \psi_1, \psi_2 > 0 \). If that were not the case, then arbitrage opportunities would exist.

The \( \psi \)'s are called state prices. They reflect how much one would pay for a security that has a payoff in only one state. We can normalize these state prices to compute a probability measure over the state space. This is called the risk neutral measure.

\[ \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} = \begin{pmatrix} \frac{\psi_1}{\psi_1 + \psi_2} \\ \frac{\psi_2}{\psi_1 + \psi_2} \end{pmatrix} \]  

(21)

2.5 - Risk Neutral Pricing

If we look at (20) for just the risky asset we have

\[ p[0] = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}^T \begin{pmatrix} p^+ [1] \\ p^- [1] \end{pmatrix} = (\psi_1 + \psi_2) \begin{pmatrix} q_1 \\ q_2 \end{pmatrix}^T \begin{pmatrix} p^+ [1] \\ p^- [1] \end{pmatrix} \]  

(22)

For the riskfree asset we have

\[ 1 = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}^T \begin{pmatrix} 1 + r_f \\ 1 + r_f \end{pmatrix} \Rightarrow \frac{1}{1 + r_f} \]  

(23)

Combining these results gives risk neutral pricing

\[ p[0] = \frac{1}{1 + r_f} \hat{E}[p[1]] \]  

(24)
2.6 - Alternative Pricing Models

The simplest stochastic pricing situation is one in which we pay a price of $p$ today in return for an uncertain cash flow $c$ a fixed period in the future.

Discounted Expected Value

\[ p = \frac{\mathbb{E}[c]}{1 + r_f} \]  \hspace{1cm} (25)

This approach typically produces a price which is too high. We either have to adjust the numerator downward or the denominator upward in order to risk adjust this computation.

CAPM Pricing

\[ p = \frac{\mathbb{E}[c]}{1 + r_f + \beta (\mathbb{E}[r_m] - r_f)} \]  \hspace{1cm} (26)

The value of $(1 + r_f) + \beta_c (\mathbb{E}[r_m] - r_f)$ is the risk adjusted rate of return.

CE Form of CAPM Pricing

We can derive this result from the first CAPM pricing form by noting that

\[ \beta = \frac{\mathbb{Cov}[c / p - 1, r_m]}{\mathbb{E}^2[\sigma_m]} = \frac{\mathbb{Cov}[r_m, c]}{\mathbb{E}^2[\sigma_m]} \]  \hspace{1cm} (27)

Substituting in and solving for the price gives

\[ p = \frac{\mathbb{E}[c] - \mathbb{Cov}[r_m, c] (\mathbb{E}[r_m] - r_f) / \sigma_m^2}{1 + r_f} \]  \hspace{1cm} (28)

The value of $\mathbb{E}[c] - \mathbb{Cov}[r_m, c] (\mathbb{E}[r_m] - r_f) / \sigma_m^2$ is the certainty equivalent of $c$.

Log Optimal Pricing

\[ p = \frac{\mathbb{E}[c]}{1 + r^*} \]  \hspace{1cm} (29)

Here the rate of return of the log optimal portfolio is used as a risk adjusted rate of return.
Risk Neutral Pricing

\[ p = \frac{\mathbb{E}[c]}{1 + r_f} \]  \hspace{1cm} (30)

In this final case the underlying probability measure is adjusted so that the discounted expected value equals the price.