AMS-512 Capital Markets and Portfolio Theory

Mid-Term Examination Solution — Spring 2018

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Question 1. (30 points)

You are given three stocks, $i = 1, 2, 3$. Each stock's return can be modeled by the capital asset pricing model (CAPM) :

$$r_i - r_f = \beta_i (m - r_f) + \epsilon_i$$

Let

r_i	=	the return of stock i
r_f	=	the risk-free rate
β_i	=	the beta (market exposure) of stock i
m	=	the market return
ϵ_i	=	the error term for stock i

Let $r_f = 0.01$, $\beta = \{0.8, 1.0, 1.2\}$, $m = 0.08$, $\sigma_m = 0.01$ and $\sigma_\epsilon = \{0.008, 0.006, 0.01\}$.

(a) What is the mean vector $\boldsymbol{\mu}$ for a mean-variance portfolio problem?

(b) What is the covariance matrix $\boldsymbol{\Sigma}$ for a mean-variance portfolio problem?

(c) What is the optimal mean-variance portfolio? (*Hint*: Recall that this answer has a particularly simple form for the CAPM.)

Solution

The parameters given are:

In[ ]:=

```
nRiskFree = 0.01;
vnBeta = {0.8, 1.0, 1.2};
nMeanMkt = 0.08;
nSdevMkt = 0.01;
vnSdevEps = {0.008, 0.006, 0.01};
```

(a) Rearranging terms in the CAPM and taking expectations yields

$$\mu_i = r_f + \beta_i(\mu_m - r_f)$$

```
In[∘]:= vnMean = nRiskFree + vnBeta (nMeanMkt - nRiskFree);
        Print["μ = ", MatrixForm[vnMean]]
```

$$\mu = \begin{pmatrix} 0.066 \\ 0.08 \\ 0.094 \end{pmatrix}$$

(b) The variance components from market exposure and errors are uncorrelated; therefore,

$$\boldsymbol{\Sigma} = \sigma_m{}^2\, \boldsymbol{\beta}\, \boldsymbol{\beta}^T + \begin{pmatrix} \sigma_{\epsilon_1}{}^2 & 0 & 0 \\ 0 & \sigma_{\epsilon_2}{}^2 & 0 \\ 0 & 0 & \sigma_{\epsilon_3}{}^2 \end{pmatrix}$$

```
In[∘]:= vnCovariance =
          nSdevMkt^2 KroneckerProduct[vnBeta, vnBeta] + DiagonalMatrix[vnSdevEps];
        Print["Σ = ", MatrixForm[vnCovariance]]
```

$$\Sigma = \begin{pmatrix} 0.008064 & 0.00008 & 0.000096 \\ 0.00008 & 0.0061 & 0.00012 \\ 0.000096 & 0.00012 & 0.010144 \end{pmatrix}$$

(c) The solution of the optimal portfolio has the following simple form:

$$(\mathbf{x}_{\text{opt}})_i \propto \frac{\beta_i}{\sigma_{\epsilon_i}{}^2}$$

```
In[∘]:= vnPortfolio = #/Total[#] &[vnBeta / vnSdevEps^2];
        Print["x_opt = ", MatrixForm[vnPortfolio]]
```

$$x_{opt} = \begin{pmatrix} 0.239107 \\ 0.53135 \\ 0.229543 \end{pmatrix}$$

Question 2. (30 points)

A collection of assets can be modeled by a multi-factor model as follows:

$$\mathbf{r} = \mathbf{B}\,\mathbf{f} + \boldsymbol{\epsilon}$$

There are $n = 500$ assets and $m = 8$ factors. The factors are uncorrelated and have standard deviations of 1.

(a) What are the dimensions of the matrix **B**?

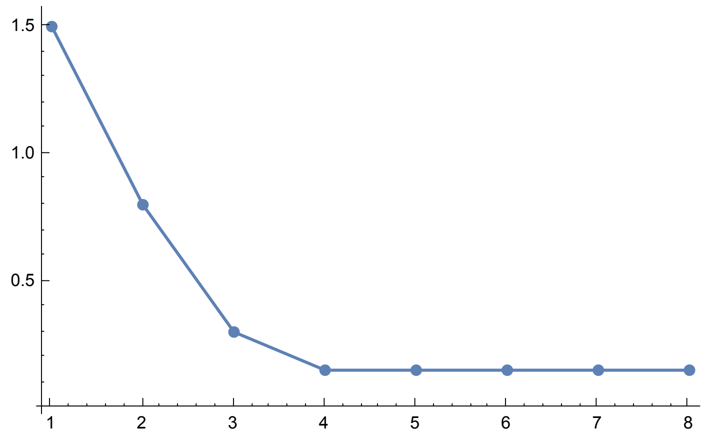
(b) Let **D** denote the covariance matrix of the error terms so the covariance matrix $\boldsymbol{\Sigma}$ of the assets is

$$\boldsymbol{\Sigma} = \mathbf{B}^T\,\mathbf{B} + \mathbf{D}$$

Show that $\boldsymbol{\Sigma}$ is positive definite.

(c) A plot of the spectrum of a raw correlation matrix of returns appears below. Based on this, how many factors would you consider using in a factor model? Explain your answer.

Out[∘]=



Solution

(a) The factor loading matrix $\mathbf{B}$ is a 500×8 matrix.

(b) A matrix $\mathbf{\Sigma}$ is positive definite iff $\mathbf{x}^T \mathbf{\Sigma}\, \mathbf{x} > 0 \; \forall \; \mathbf{x} \neq \mathbf{0}$. We can expand $\mathbf{x}^T \mathbf{\Sigma}\, \mathbf{x}$ as follows:

$$\mathbf{x}^T \mathbf{\Sigma}\, \mathbf{x} = \mathbf{x}^T (\mathbf{B}\, \mathbf{B}^T + \mathbf{D})\, \mathbf{x} = (\mathbf{B}^T \mathbf{x})^T (\mathbf{B}^T \mathbf{x}) + \mathbf{x}^T \mathbf{D}\, \mathbf{x}$$

Now $\mathbf{y} = \mathbf{B}^T \mathbf{x}$ is a vector but $\mathbf{B}$ is clearly not of full rank; hence, there are $\mathbf{x} \neq \mathbf{0}$ in the null space of $\mathbf{B}$ such that $\mathbf{y} = \mathbf{0}$. Therefore, $\mathbf{y}^T \mathbf{y} = \sum_i y_i^2 \geq \mathbf{0}$.

The diagonal matrix $\mathbf{D}$ is of full rank; therefore for $\mathbf{x} \neq \mathbf{0}$, we have $\mathbf{x}^T \mathbf{D}\, \mathbf{x} = \sum_i x_i^2 / \sigma_{\epsilon_i}{}^2 > 0$.

The expression $(\mathbf{B}^T \mathbf{x})^T (\mathbf{B}^T \mathbf{x}) + \mathbf{x}^T \mathbf{D}\, \mathbf{x}$, for $\mathbf{x} \neq \mathbf{0}$ is the sum of a non-negative and positive number and is itself positive.

(c) We don't have the information to use the M-P distribution to denoise the correlations. (We know $N = 500$ but have no idea what T is.) If you approached the problem from a denoising perspective, then you will receive credit if you described the process correctly. Here however, we must rely on the pattern evidenced by the spectrum. The answer is simple: The first 3 eigenvalues have unique values while the remaining 7 are undifferentiated. A 3-factor model would appear to be a reasonable choice. However, if you chose a 4-factor model you received full credit.

Question 3. (10 points)

Assume that you have one unit of capital to invest and can go both long and short on stock positions. The tangent portfolio is

$$\mathbf{x}_{\text{tangent}} = \left(\frac{1}{\mathbf{1}^T \mathbf{\Sigma}^{-1} (\boldsymbol{\mu} - \mathbf{1}\, r_f)} \right) \mathbf{\Sigma}^{-1} (\boldsymbol{\mu} - \mathbf{1}\, r_f)$$

Assuming that the stocks are all uncorrelated, write a simple proportional form for x_i the amount allocated to stock i in the tangent portfolio.

Solution

If the asset returns are uncorrelated, then $\mathbf{\Sigma}$ and its inverse are diagonal matrices; hence, we have the simple proportional form:

$$\left(\mathbf{x}_{\text{tangent}}\right)_i \propto \frac{\mu_i - r_f}{\sigma_i^2}$$

where σ_i^2 is the i^{th} diagonal element of $\mathbf{\Sigma}$.

Question 4. (30 points)

Describe the following *Mathematica* statements. For example, if x is a vector of numbers, then for

```
Select[x, √# > 10&]
```

the answer would be “Selects the elements of x whose square roots are greater than 10."

(a) If **x** and **y** are vectors of numbers of the same length, describe the following:

$$\mathbf{x}^{\mathbf{y}} + 2$$

(b) If **x** is a vector of numbers, describe the result of the following:

```
#/Total[#] &[Sin /@ x]
```

(c) You have a vector of strings **s** which contains the tickers of a number of stock. How can you use FinancialData[ ] to retrieve their names into a vector of strings **n**?

Solution

(a) Take each element in **x** and raise it to the corresponding power in **y**, then add 2 to each element of the result. For example,

```
In[∘]:= x = {1, 2, 3};
        y = {2, 4, 6};
        x^y + 2
Out[∘]= {3, 18, 731}
```

(b) Take the sine of each element of a vector of numbers **x** and then normalize them so that they add to 1. For example,

```
In[∘]:= # / Total[#] &[Sin /@ {√π, π / 2, π^2}]
```

$$\text{Out}[\circ]= \left\{\frac{\text{Sin}[\sqrt{\pi}]}{1+\text{Sin}[\sqrt{\pi}]+\text{Sin}[\pi^2]}, \frac{1}{1+\text{Sin}[\sqrt{\pi}]+\text{Sin}[\pi^2]}, \frac{\text{Sin}[\pi^2]}{1+\text{Sin}[\sqrt{\pi}]+\text{Sin}[\pi^2]}\right\}$$

Actually, the /@ is unnecessary. Sin[ ] ilas the attribute Listable, *i.e.*, it “pushes” itself down to the lowest level of it argument. First, the attributes of Sin[ ], and the code without /@ (Map[ ]).

```
In[∘]:= Attributes[Sin]
Out[∘]= {Listable, NumericFunction, Protected}
```

Thus, we could have also coded:

In[∘]:=
```
#/Total[#] &[Sin[{√π, π/2, π^2}]]
```

Out[∘]= $\left\{\frac{\text{Sin}[\sqrt{\pi}]}{1+\text{Sin}[\sqrt{\pi}]+\text{Sin}[\pi^2]}, \frac{1}{1+\text{Sin}[\sqrt{\pi}]+\text{Sin}[\pi^2]}, \frac{\text{Sin}[\pi^2]}{1+\text{Sin}[\sqrt{\pi}]+\text{Sin}[\pi^2]}\right\}$

Note that not all functions are Listable, but many of the common ones are such as the trig and log functions.

(c) The code below assigns to **n** the names of the ticker symbols in **s**. For example, for **s** = {“IBM”, ”GE”, ”MSFT”, ”X”}

In[43]:=
```
s = {"IBM", "GE", "MSFT", "X"};
n = FinancialData[#, "Name"] & /@ s;
Print["s = ", MatrixForm[s]]
```

$$s = \begin{pmatrix} \text{IBM} \\ \text{GE} \\ \text{MSFT} \\ \text{X} \end{pmatrix}$$