Modeling Risk in Arbitrage Strategies Using Hierarchical Mixture Models

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Abstract

A general class of models is developed that captures the state-dependent character of arbitrage-based hedge fund strategies. Using merger arbitrage as an example, and specific instance is described and its MLE is illustrated.

This modeling approach provides a better description of real hedge fund returns in a framework that is more general and more easily extensible than other alternatives in the open literature.

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Benefits

- Economically Rational Foundation
- Improved Signal-To-Noise
- Systematic Hedge $\Rightarrow$ Bet Independence
- Decomposition $\Rightarrow$ Simpler Models

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Dangers

- Inefficiencies Small and Transient
- Leverage Required $\Rightarrow$ Hedge Instability
- Model Failure $\Rightarrow$ Catastrophic Returns

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Net Result

- Strategies produce stable but modest returns punctuated by intervals of dramatically poor performance.
- Failure occurs when leveraged, ostensibly independent bets behave in ways that are pathological and highly correlated.

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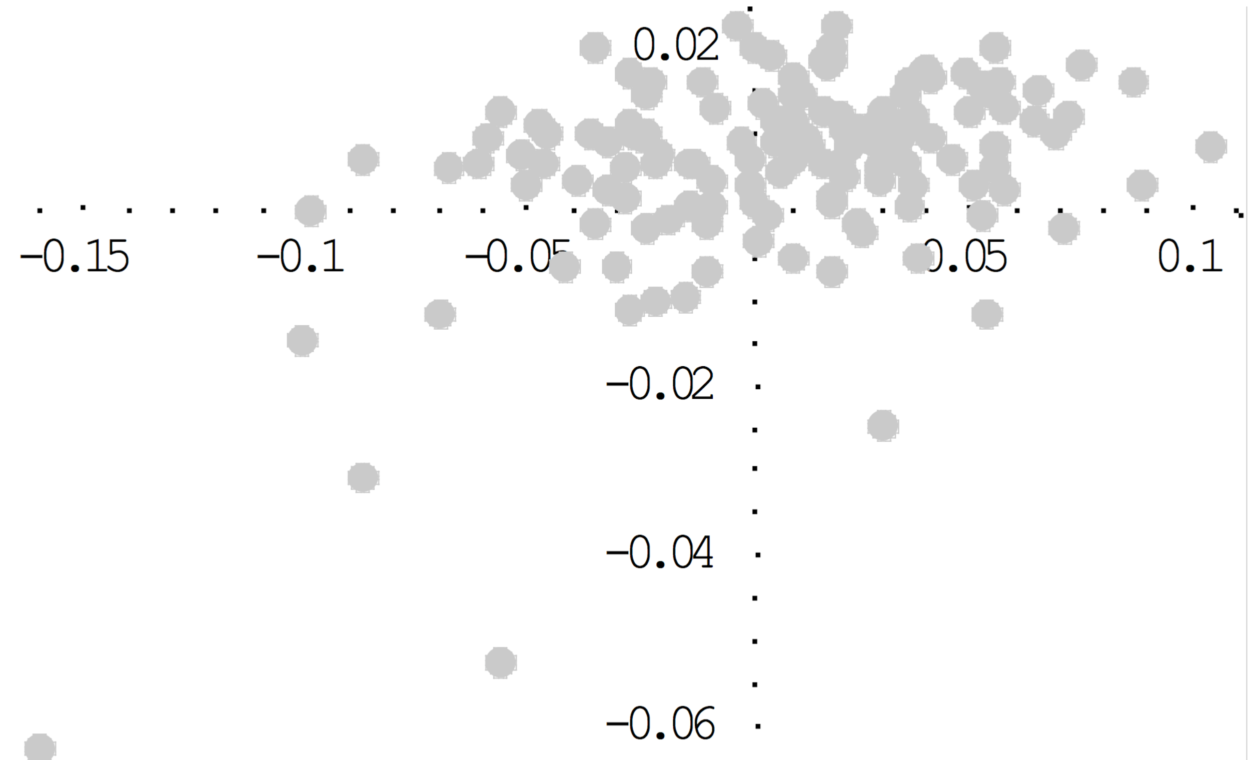
HFR Meger Arb *vs.* S &P 500

Monthly: 1990-2001



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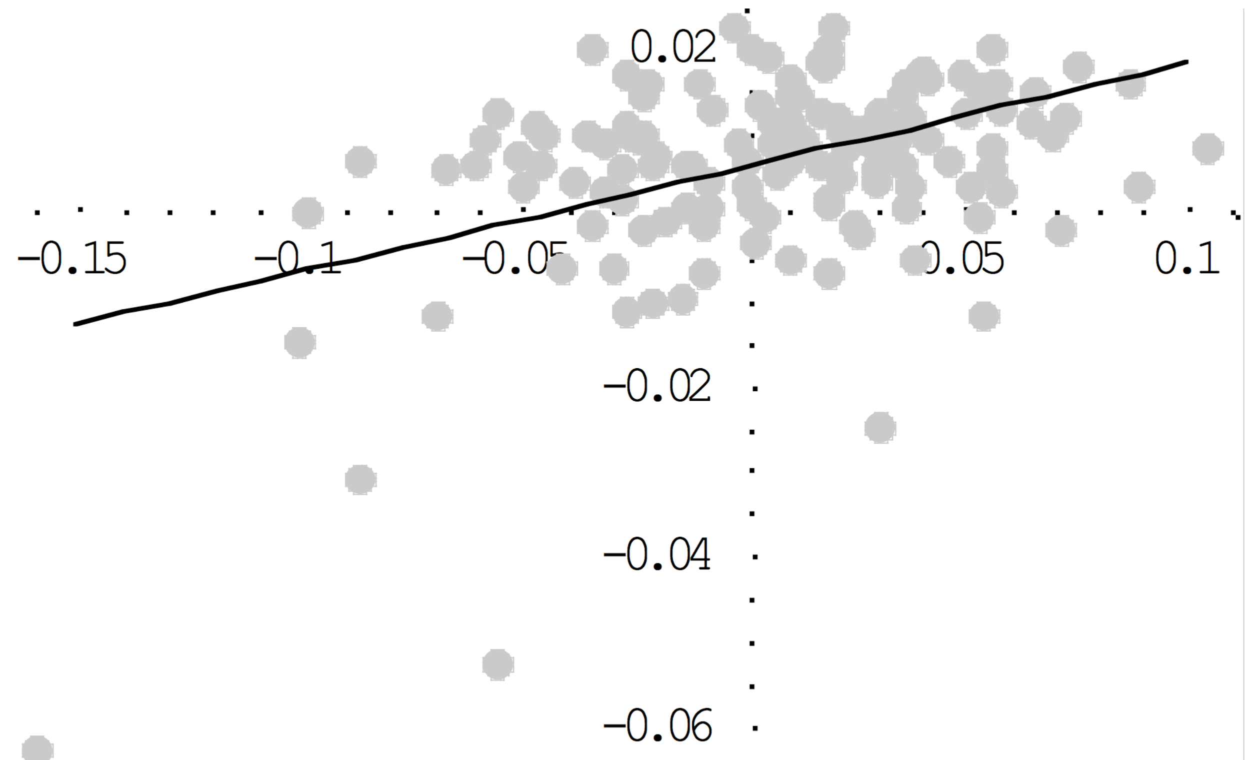
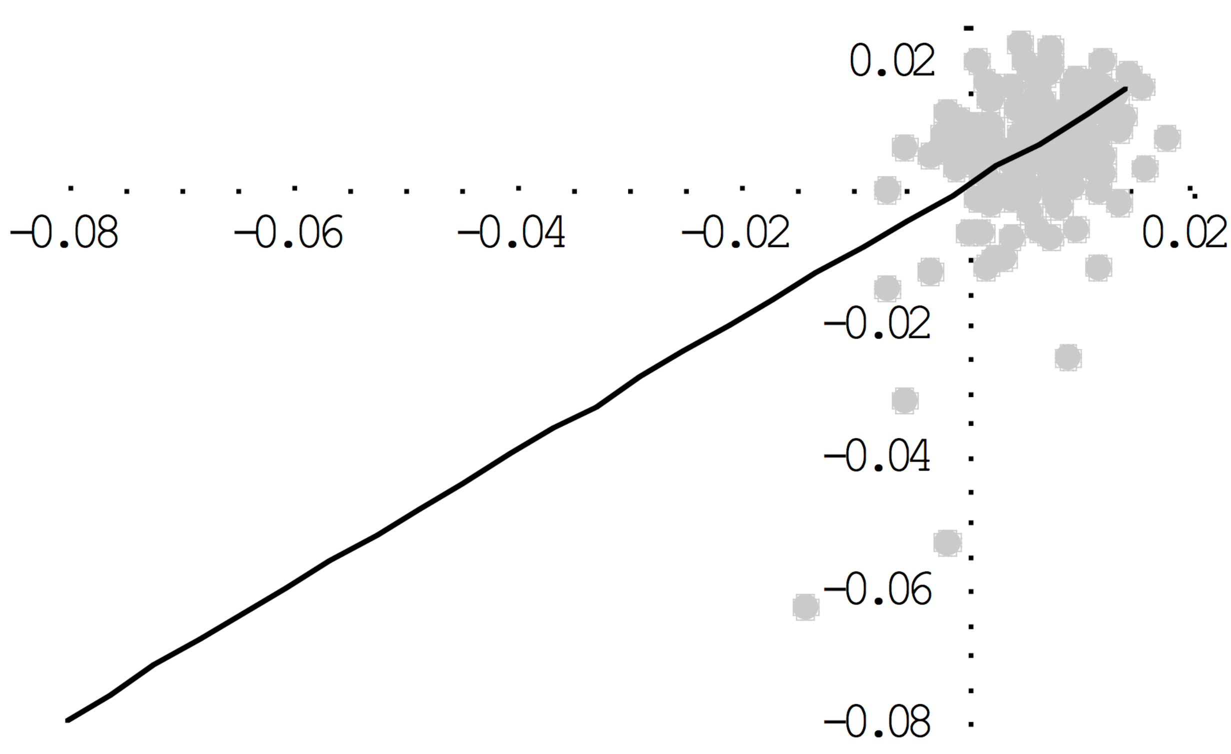


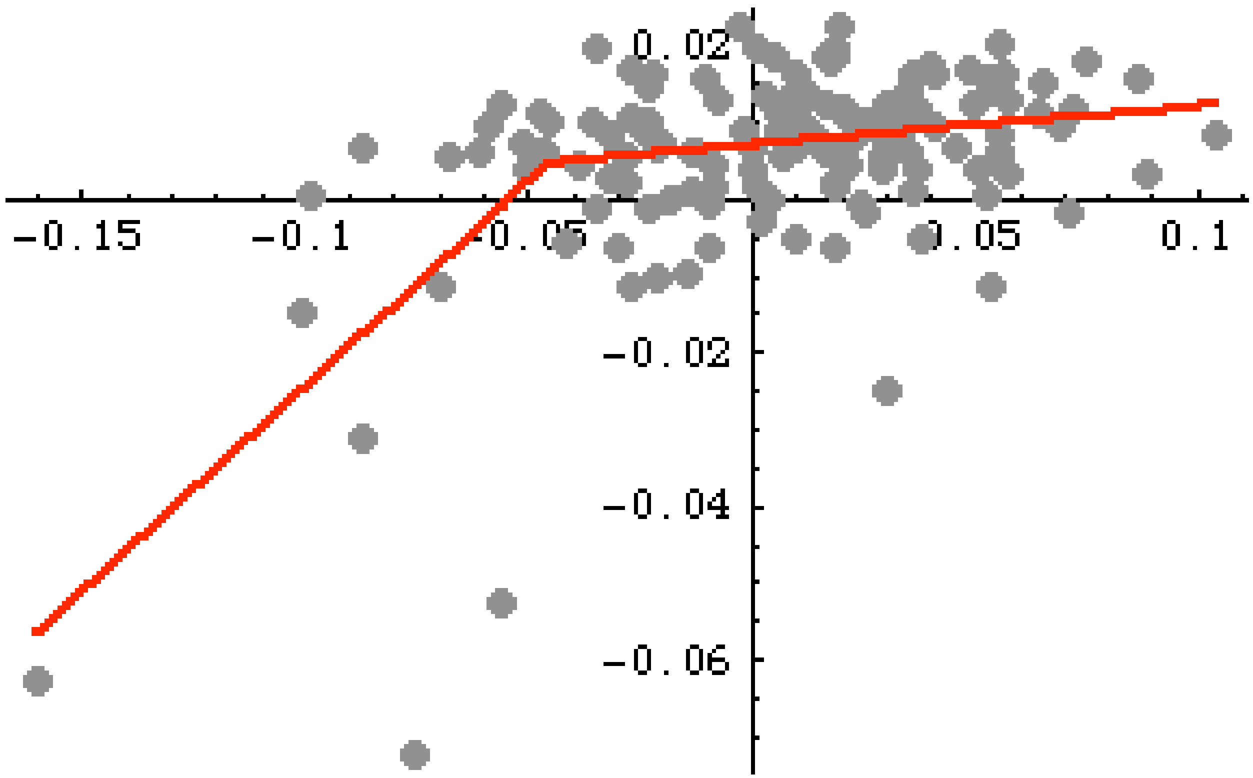
Linear Model

Forecast *vs.* Actual



Mitchel & Pulvino's Kinked Model

Extended to optimal least squares fit



Generalized r-Squared

$$r^2_{\text{generalized}} = 1 - \frac{\sum_{j=1}^{n}\left(y_j - \hat{y}_j\right)^2 \Big/ (n-k)}{\sum_{j=1}^{n}\left(y_j - \mathrm{E}[y]\right)^2 \Big/ (n-1)}$$

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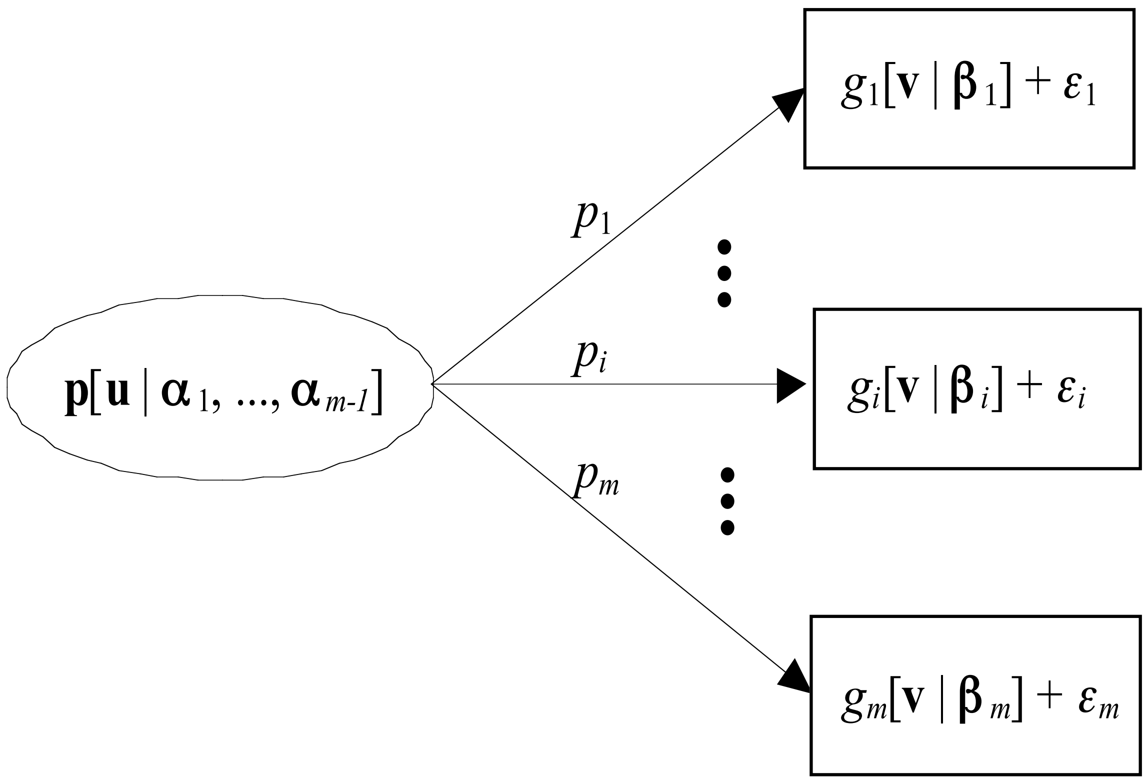
Parameterized Mixture



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PFMC Return Model

$$r = \mathbf{p}\left[\mathbf{u} \mid \boldsymbol{\alpha}_1, \ldots, \boldsymbol{\alpha}_{m-1}\right]^T \left(\mathbf{g}\left[\mathbf{v} \mid \boldsymbol{\beta}_1, \ldots, \boldsymbol{\beta}_m\right] + \boldsymbol{\phi}\left[\boldsymbol{\varepsilon} \mid \boldsymbol{\Omega}\right]\right)$$

$$f\left[r, \mathbf{u}, \mathbf{v} \mid \Psi\right] = \sum_{i=1}^{m} p_i\left[\mathbf{u} \mid \boldsymbol{\alpha}_i\right] \phi\left[r - g_i[\mathbf{v} \mid \boldsymbol{\beta}_i] \mid \Omega_i\right]$$

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Log Likelihood

$$\Lambda = \sum_{j=1}^{n} \log\left[\sum_{i=1}^{m} p_i\left[\mathbf{u}_j \mid \boldsymbol{\alpha}_i\right]\phi\left[r_j - g_i[\mathbf{v}_j \mid \boldsymbol{\beta}_i] \;\middle|\; \Omega_i\right]\right]$$

$$\Lambda_C = \sum_{i=1}^{m}\sum_{j=1}^{n} z_{i,j}\left(\log\left[p_i\left[\mathbf{u}_j \mid \boldsymbol{\alpha}_i\right]\right] + \log\left[\phi\left[r_j - g_i[\mathbf{v}_j \mid \boldsymbol{\beta}_i] \;\middle|\; \Omega_i\right]\right]\right)$$

LMLC Model

$$\mathbf{p}\left[\boldsymbol{\alpha}^T\mathbf{u}\right] = \begin{pmatrix} p_1 = \dfrac{e^{\boldsymbol{\alpha}^T\mathbf{u}}}{1+e^{\boldsymbol{\alpha}^T\mathbf{u}}} \\ p_2 = \dfrac{1}{1+e^{\boldsymbol{\alpha}^T\mathbf{u}}} \end{pmatrix}$$

$$g_i[\mathbf{v}\,|\,\boldsymbol{\beta}_i] = \boldsymbol{\beta}_i^T\mathbf{v}, \qquad i = 1,\ 2$$

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MLE

$$\Lambda_C \propto \sum_{j=1}^{n}\left(z_{1,j}\log\left[\frac{e^{\boldsymbol{\alpha}^T\mathbf{u}_j}}{1+e^{\boldsymbol{\alpha}^T\mathbf{u}_j}}\right] + z_{2,j}\log\left[\frac{1}{1+e^{\boldsymbol{\alpha}^T\mathbf{u}_j}}\right]\right) -$$

$$\tfrac{1}{2}\sum_{j=1}^{n}\left(z_{1,j}\left(\log[\sigma_1^2] + \frac{(r_j - \boldsymbol{\beta}_1{}^T\mathbf{v}_j)^2}{\sigma_1^2}\right) + z_{2,j}\left(\log[\sigma_2^2] + \frac{(r_j - \boldsymbol{\beta}_2{}^T\mathbf{v}_j)^2}{\sigma_2^2}\right)\right)$$

Estimated using the EM Algorithm

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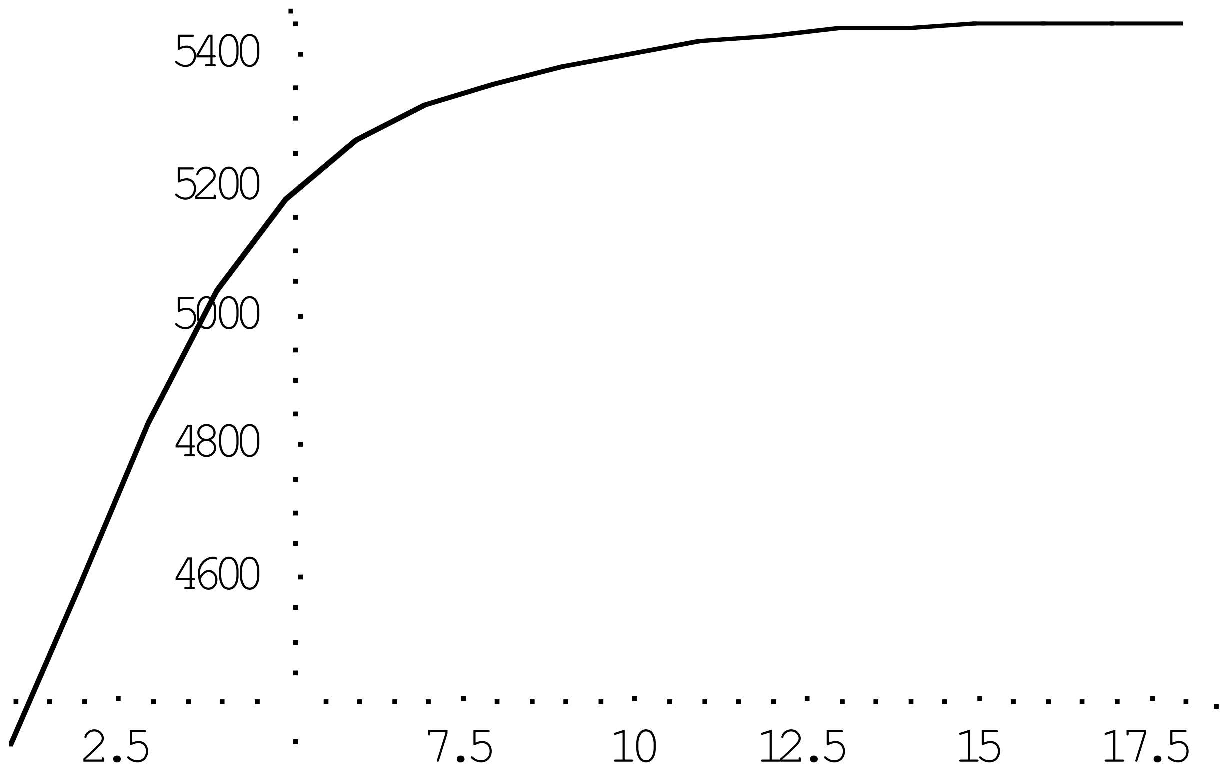
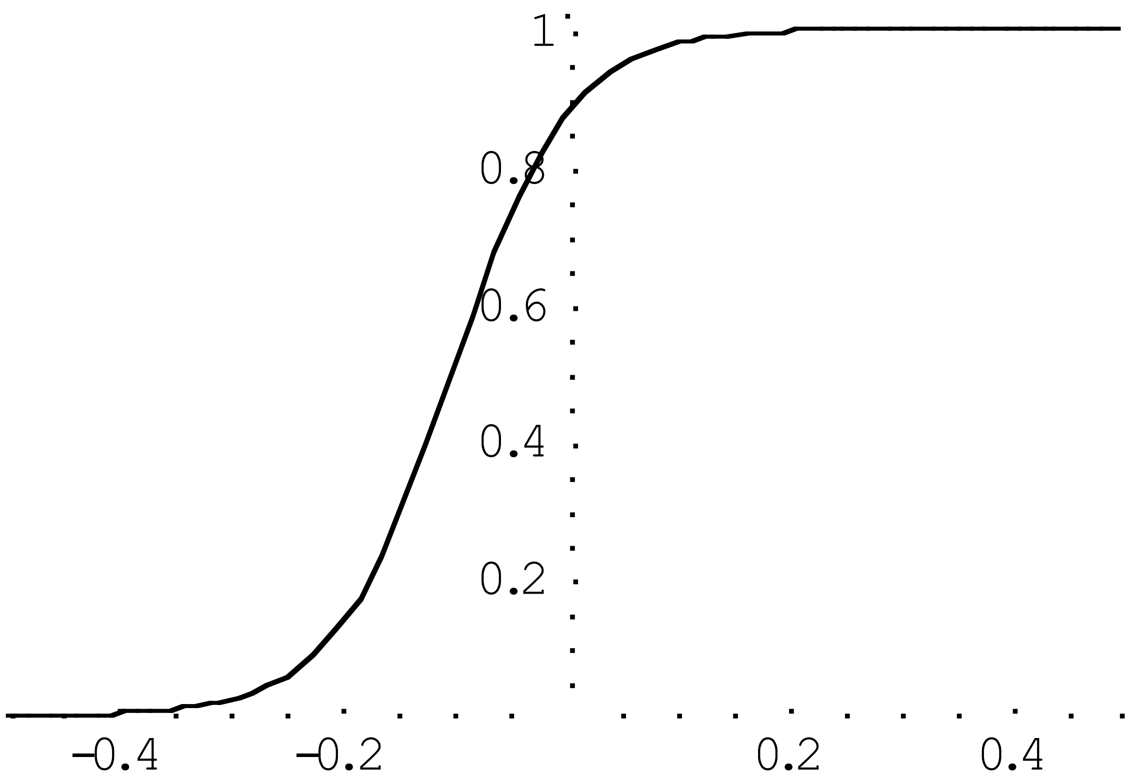
Λ-Convergence



Mixture Function



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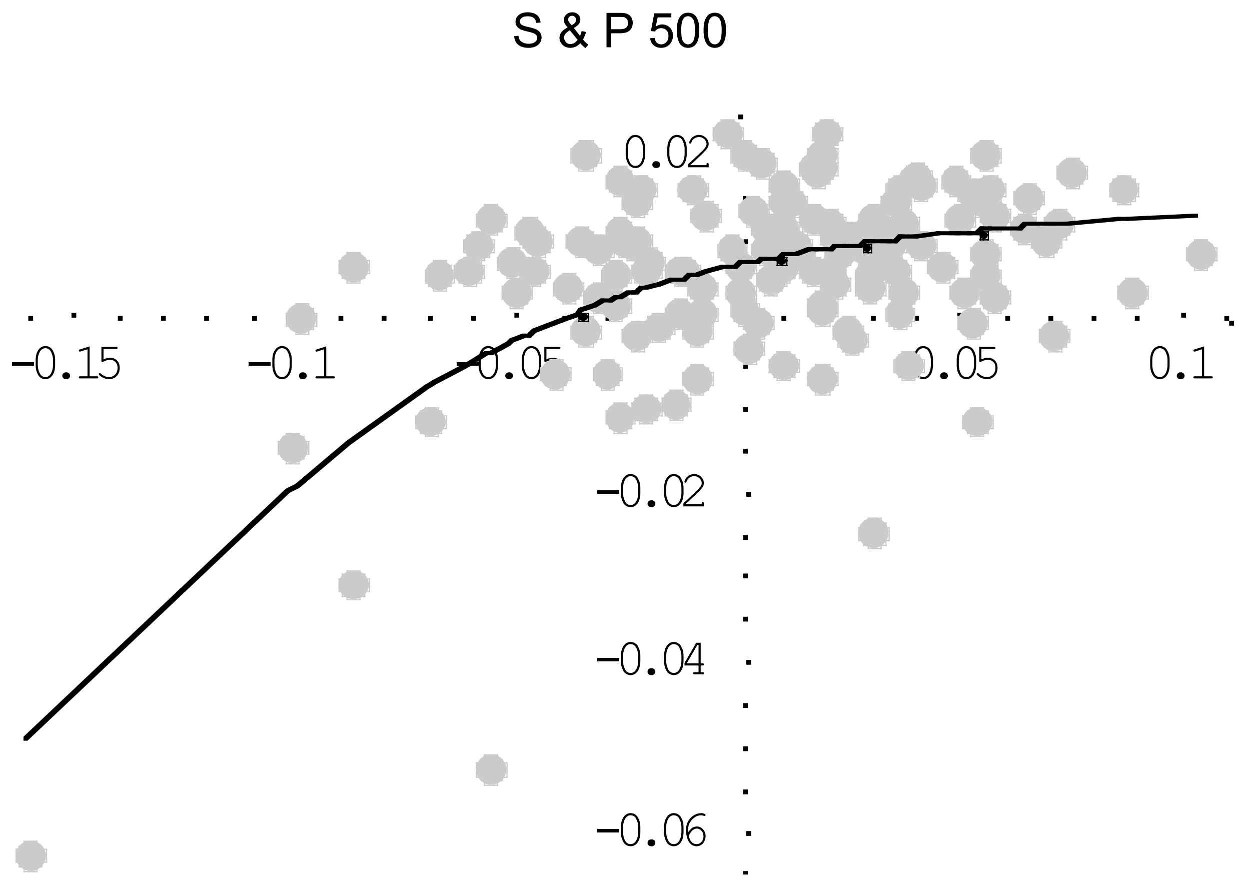
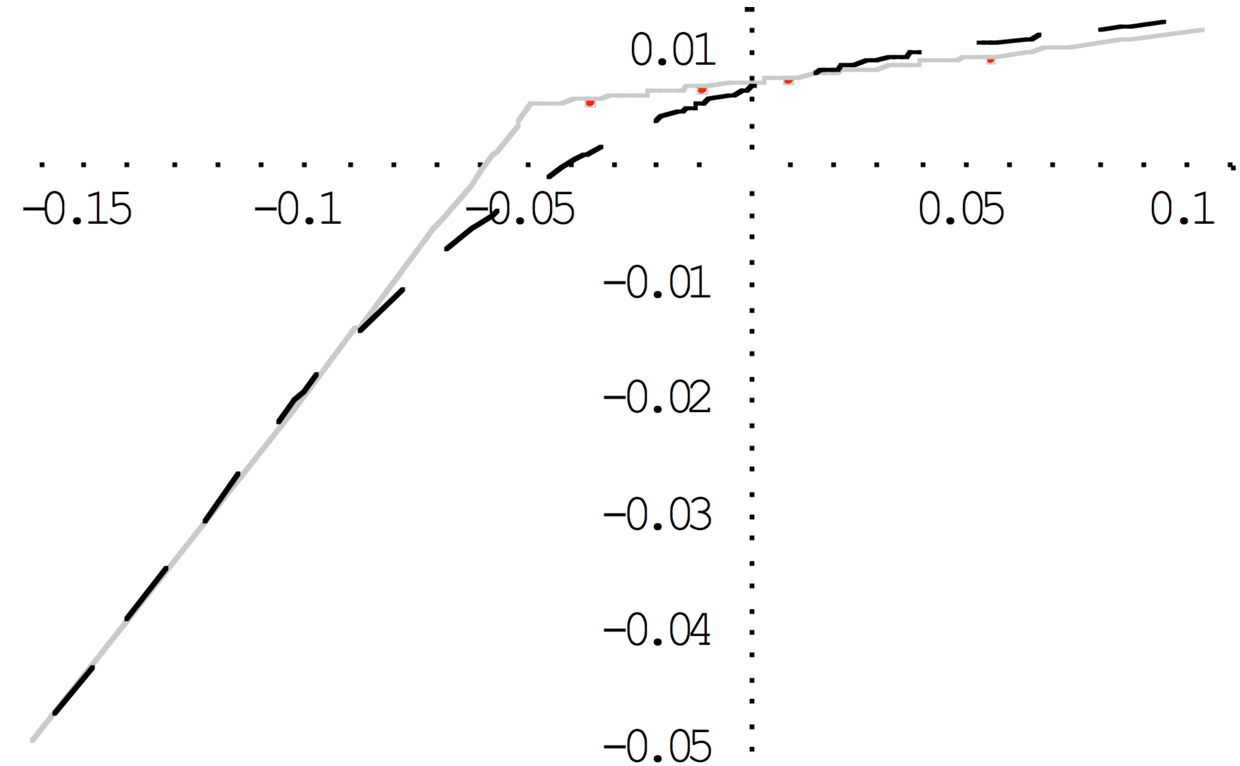
LMLC Single Factor Model

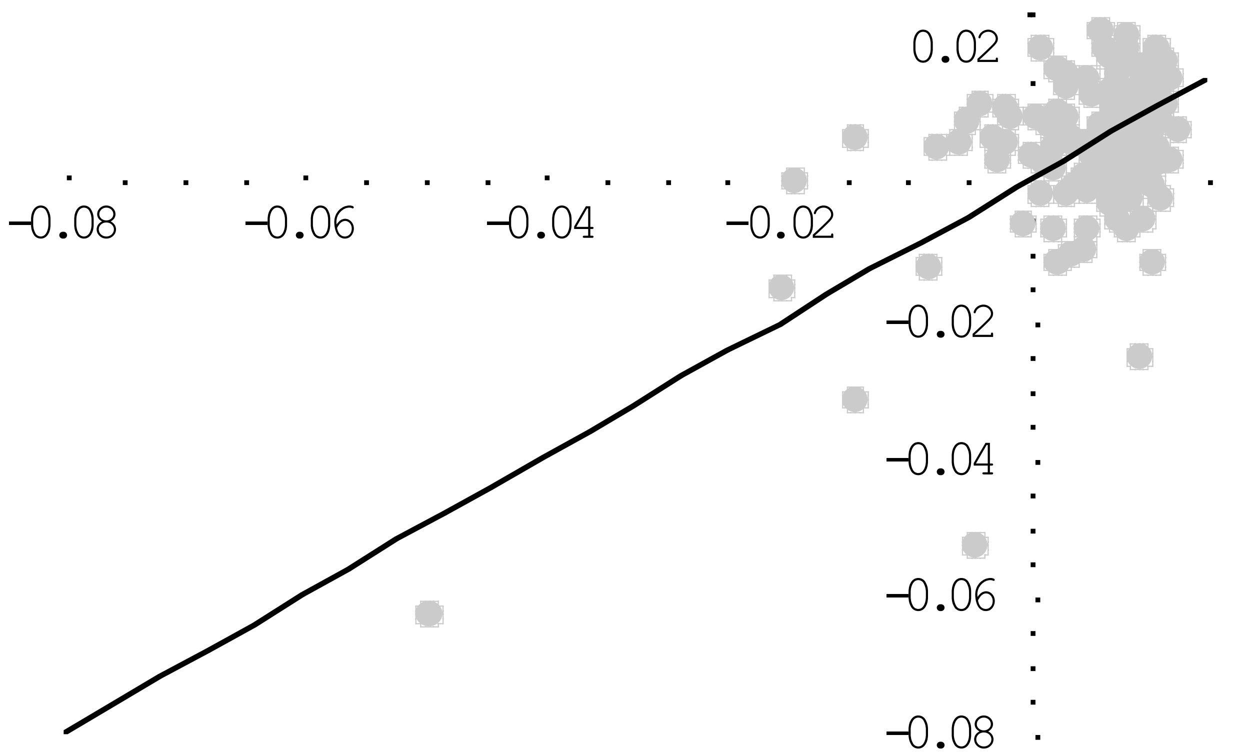


Single Factor Comparison



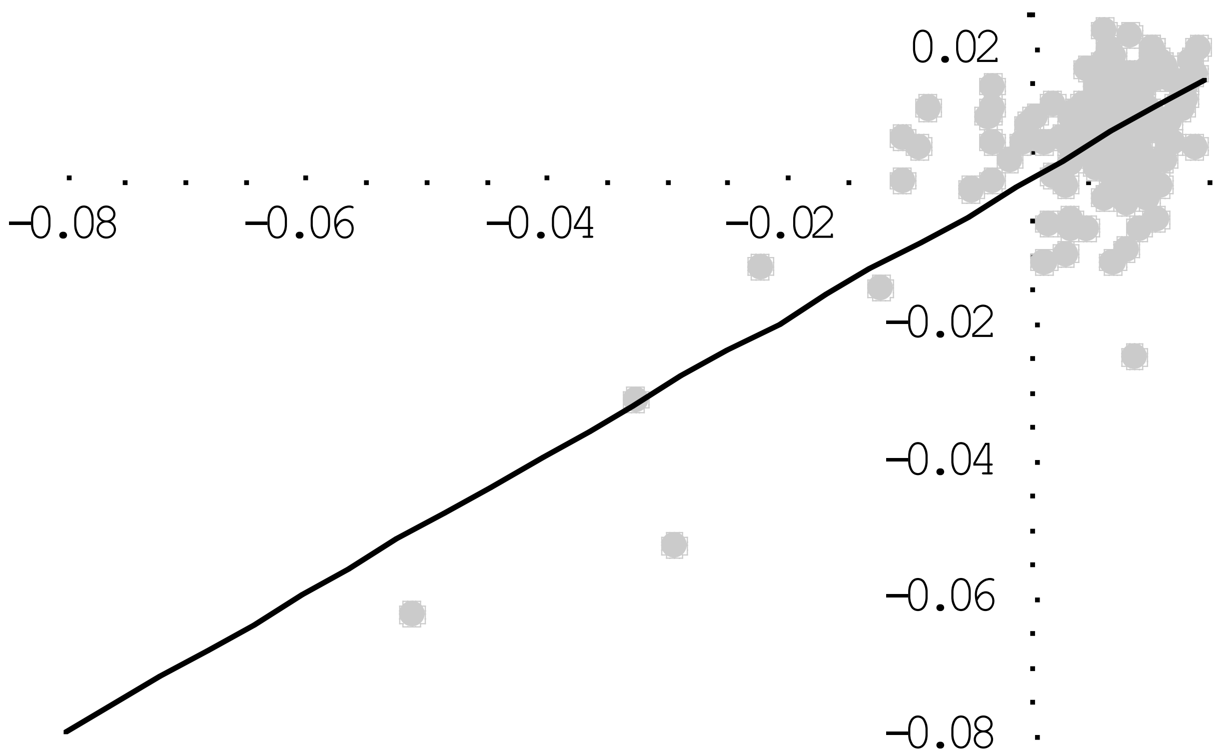
LMLC Single Factor Errors

Forecast *vs.* Actual



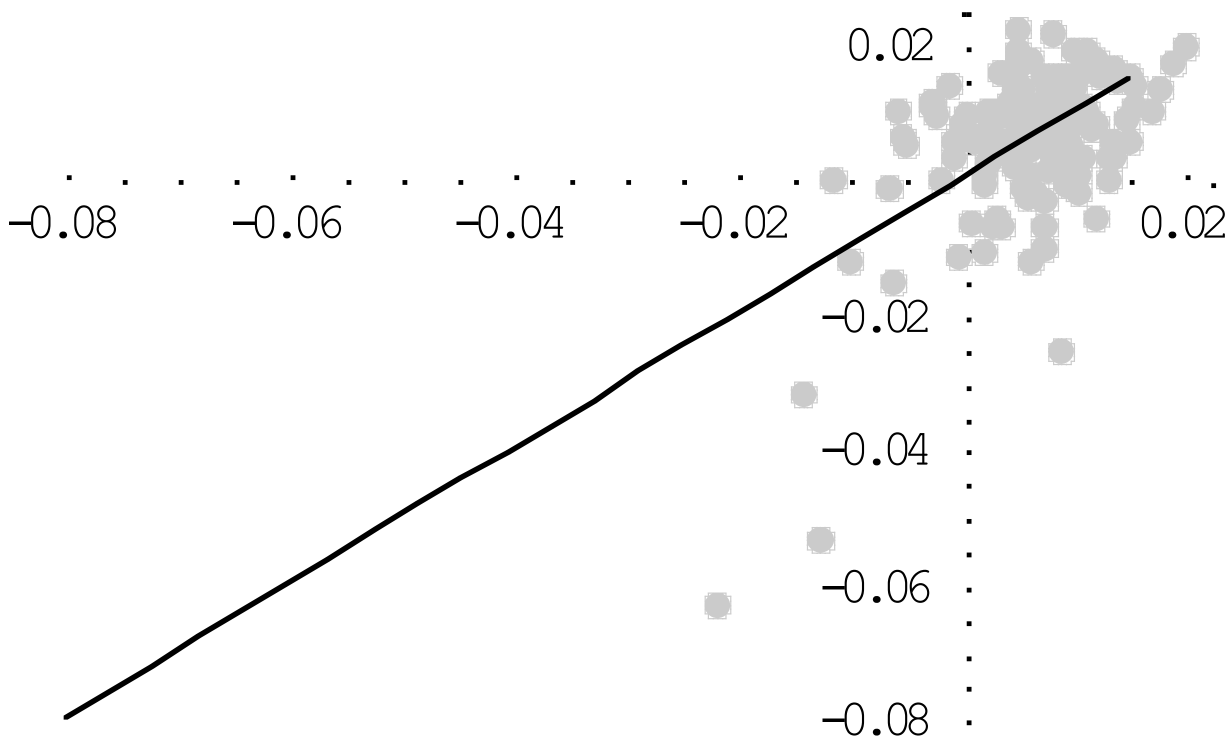
LMLC Multi-Factor Errors

S & P 500, Lagged S & P 500, Credit Spreads



Linear Multi-Factor Errors

Fit per Mitchell [1999], Rahl [2000] and Schneeweis [2000]



Model Performance Summary

Model	$r^2_{\text{generalized}}$
Single Factor Linear	0.188
Multi-Factor Linear	0.298
Kinked Single Factor	0.334
Single Factor LMLC	0.298
Multi-Factor LMLC	0.428

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