Reduction of average-cost Markov Decision Processes to discounting under an accessibility condition

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Plan of the talk

1. Definitions
2. Review: complexity of discounted MDPs
3. Review: complexity of average-cost MDPs
4. Reducing average-cost MDPs to discounting
   ▶ Complexity of policy iteration
   ▶ Existence of optimal policies - infinite state spaces
Markov Decision Process (MDP): defined by \((X, A(\cdot), p, c)\) where

1. \(X\) - state space
2. \(A(x)\) - sets of actions available at \(x \in X\)
3. \(p(y|x, a)\) - transition probabilities, where
   - \(x\) - current state
   - \(a\) - current action
   - \(y\) - next state
4. \(c(x, a)\) - one-step costs

**Assume:** \(X\) is **discrete**, \(A(x)\) is **finite** \(\forall x \in X\).
Definitions: Policies

Policy $\pi$ - history-dependent and randomized in general.

$\Pi := \text{set of all policies.}$

Stationary policy $\phi$: selects action $\phi(x) \in A(x)$ whenever the state is $x \in X$.

$\mathcal{F} := \text{set of all stationary policies.}$

For $\pi \in \Pi$ & initial $x \in X$, the average cost is

$$w^\pi(x) := \limsup_{N \to \infty} \frac{1}{N} \mathbb{E}_x^\pi \sum_{n=0}^{N-1} c(x_n, a_n);$$

for $\beta \in [0, 1)$ the $\beta$-discounted cost is

$$v_\beta^\pi(x) := \mathbb{E}_x^\pi \sum_{n=0}^{\infty} \beta^n c(x_n, a_n).$$
Definitions: Optimality

\( \pi^* \in \Pi \) is **average-cost optimal** if

\[
    w^{\pi^*}(x) = \inf_{\pi \in \Pi} w^\pi(x) \quad \forall x \in X
\]

and **\( \beta \)-discount optimal** if

\[
    v^{\pi^*}_\beta(x) = \inf_{\pi \in \Pi} v^\pi_\beta(x) \quad \forall x \in X.
\]

Main questions:

1. When do (stationary) optimal policies exist?
2. How can optimal policies be computed (and how quickly)?
Computing optimal policies

*Main methods:*

1. **Value Iteration**
   - discounted: Shapley (1953)
   - undiscounted: Bellman (1957)
   - average-cost: White (1963)

2. **Policy Iteration**
   - discounted & average-cost: Howard (1960)

3. **Linear Programming Algorithms via LP formulation**
   - discounted: D’Epenoux (1963)
   - average-cost: de Ghellinck (1960) and Manne (1960); Denardo and Fox (1968), Hordijk and Kallenberg (1979, 1980), Kallenberg (1983)
   - Wolfe and Dantzig (1962)

*Focus of talk: *Policy Iteration & Simplex Method*
Definitions: Complexity of algorithms

\[ m := \text{number of state-action pairs } (x, a), \ x \in \mathbb{X}, \ a \in A(x) \]

Two classes of “efficient” algorithms:

- **weakly polynomial**: number of arithmetic operations needed is bounded above by a polynomial in \( m \) & the bit-size \( L \) of the input data;

- **strongly polynomial**: number of arithmetic operations needed is bounded above by a polynomial in \( m \) only.
Complexity results: Discounted costs (fixed $\beta$)

Value Iteration:
- weakly polynomial - Tseng (1990)
- not strongly polynomial - Feinberg and H. (2014)

Howard’s Policy Iteration:
- weakly polynomial - Meister and Holzbaur (1986)

LP Algorithms:

Many modified policy iteration algorithms are not strongly polynomial - Feinberg, H., and Scherrrer (2014).
- Puterman and Shin’s (1978) algorithm, Bertsekas and Tsitsiklis’s (1996) $\lambda$-policy iteration
**Complexity results:** Discounted costs - particular models

**Simplex:** strongly polynomial, regardless of $\beta$, for:

- **deterministic MDPs** - Post and Ye (2013) (Dantzig’s rule), sharper bound by Hansen, Kaplan, and Zwick (2014)

- **controlled random walks** (e.g. M/M/1 queues) - Zadorojniy, Even, and Schwartz (2009) & Even and Zadorojniy (2012) (Gass-Saaty rule)
**Complexity results:** Average costs - particular models

**Simplex:** *strongly polynomial* for

- controlled random walks - Zadorojniy, Even and Schwartz (2009), Even and Zadorojniy (2012) (Gass-Saaty rule)
- problems with a state \( \ell \) that’s reached under any action with probability at least \( \alpha > 0 \) - Feinberg and H. (2013) (Dantzig’s rule)

**Howard’s policy iteration:** *strongly polynomial* for

- problems with a state \( \ell \) that’s reached under any action with probability at least \( \alpha > 0 \) - Feinberg and H. (2013)
- problems where the hitting time to a state \( \ell \) is uniformly bounded in starting state & policy - Akian and Gaubert (2013)
  - shown for Hoffman and Karp’s (1966) algorithm for mean-payoff games
Methods for studying complexity of average costs

Two approaches:

2. Red**uction** to discounted problem
   - Feinberg and H. (2013) - Ross’s (1968a,b) transformation
   - Akian and Gaubert (2013) - non-linear Perron-Frobenius theory

**Akian and Gaubert’s (2013) transformation**: generalization of Ross’s (1968a,b) transformation.

- see also Gubenko and Štadland (1975), Dynkin and Yushkevich (1979).
Rest of the talk

1. Sufficient conditions for & implications of Akian and Gaubert’s (2013) hitting time assumption;
2. Their reduction for MDPs without non-linear Perron-Frobenius theory;
3. Infinite $\mathbb{X}$ - obtaining existence of a stationary optimal policy.
The assumption

\( \ell \in X \) - fixed state

\( \tau_\ell := \inf\{ n \geq 1 | x_n = \ell \} = \text{hitting time to} \ \ell \)

**Assumption HT (Hitting Time)**

There's a constant \( K \) where

\[
E_x^\phi \tau_\ell \leq K < \infty \quad \forall x \in X, \phi \in \mathbb{F}.
\]

**Equivalent:** \( \exists \) bounded nonnegative function \( \xi \) on \( X \) satisfying

\[
\xi(x) \geq 1 + \sum_{y \in X \setminus \{\ell\}} p(y|x, a)\xi(y) \quad \forall a \in A(x), x \in X.
\]
Sufficient condition for Assumption HT

Assumption D

There’s a positive integer $N$ & constant $\alpha$ where

$$\mathbb{P}_x^\phi \{x_N = \ell\} \geq \alpha > 0 \quad \forall x \in X, \phi \in F.$$ 

- Special case of Hordijk’s (1974) simultaneous Doeblin condition.
- Implies

$$\mathbb{E}_x^\phi \tau_\ell \leq \frac{N}{\alpha} < \infty \quad \forall x \in X, \phi \in F.$$ 

- Ross’s (1968a,b) assumption: $N = 1$. 
Implications of Assumption HT

\( P(\phi) := \text{Markov chain corresponding to } \phi \in \mathbb{F} \)

- state \( \ell \) is *positive recurrent* \( \forall \phi \in \mathbb{F} \).
- MDP is *unichain*, i.e. \( P(\phi) \) has a single recurrent class \( \forall \phi \in \mathbb{F} \).
- If \( P(\phi) \) is aperiodic \( \forall \phi \in \mathbb{F} \),
  - each \( P(\phi) \) has a stationary distribution \( \pi(\phi) \);
  - each \( P(\phi) \) is *fast mixing* - \( \exists \) positive integer \( N \) and \( \rho < 1 \) where

\[
\sup_{B \subseteq X} \left| \sum_{y \in B} P^n(\phi)(x, y) - \sum_{y \in B} \pi(\phi)(y) \right| \leq \rho^{\lfloor n/N \rfloor} \quad \forall x \in X, \ n \geq 1;
\]

see Federgruen, Hordijk, and Tijms (1978).
- average cost \( w^\phi \) is *constant* \( \forall \phi \in \mathbb{F} \).
Reduction to discounting under Assumption HT

$\xi$ - bounded nonnegative function satisfying

$$
\xi(x) \geq 1 + \sum_{y \in X \setminus \{\ell\}} p(y|x, a)\xi(y) \quad \forall a \in A(x), \ x \in X
$$

$K$ - upper bound for $\xi$

**Step 1:** Use $\xi$ to construct MDP with state-dependent discount factors

$$
\frac{\xi(x) - 1}{\xi(x)}, \quad x \in X.
$$

**Step 2:** Construct MDP with uniform discount factor

$$
\beta := \frac{K - 1}{K}.
$$
**Step 1: State-dependent discounting - Akian and Gaubert (2013)**

1. State space $X$

2. Action sets $A(x), x \in X$

3. Transition probabilities

\[
\begin{align*}
p_{\xi}(y|x, a) :&= \begin{cases} 
\frac{1}{\xi(x)-1} p(y|x, a) \xi(y), & y \neq \ell, \\
1 - \frac{1}{\xi(x)-1} \sum_{y \neq \ell} p(y|x, a) \xi(y), & y = \ell
\end{cases}
\end{align*}
\]

4. One-step costs $c_{\xi}(x, a) := c(x, a)/\xi(x)$

5. Current state is $x \implies$ next period’s cost discounted by

\[
\gamma_{\xi}(x) := \frac{\xi(x) - 1}{\xi(x)}
\]
"Grave state" - $\bar{x} \not\in \mathbb{X}$

1. State space $\bar{\mathbb{X}} := \mathbb{X} \cup \{\bar{x}\}$

2. Action sets

   $$\bar{A}(x) := \begin{cases} A(x), & x \in \mathbb{X} \\ \{a\}, & x = \bar{x} \end{cases}$$

3. Transition probabilities

   $$\bar{p}(y|x, a) := \begin{cases} \frac{\gamma(x)}{\beta} p_\xi(y|x, a), & x, y \in \mathbb{X} \\ 1 - \frac{\gamma(x)}{\beta}, & x \in \mathbb{X}, y = \bar{x} \\ 1, & x = y = \bar{x} \end{cases}$$

4. One-step costs

   $$\bar{c}(x, a) := \begin{cases} c_\xi(x, a), & x \in \mathbb{X} \\ 0, & x = \bar{x} \end{cases}$$

5. Discount factor $\beta = (K - 1)/K$
Howard’s & simple policy iteration

Policy iteration (both discounted and average-cost):

0. Select $\phi \in \mathcal{F}$.
1. Evaluate $\phi$.
2. Improve $\phi$ if possible and go to step 1; otherwise $\phi$ is optimal.

Improvement rule $\iff$ simplex pivoting rule for LP formulation.

$\rho_{xa}$ - decision variables, $a \in A(x)$, $x \in \mathbb{X}$.

Howard’s policy iteration: For each $x$, variable $\rho_{xa}$ with most negative reduced cost enters the basis. (block pivoting)

Simple policy iteration: Variable $\rho_{xa}$ with most negative reduced cost enters basis. (Dantzig’s rule)
Correspondence of policy iterations

Lemma

A sequence of policies is generated by discounted policy iteration for the MDP \((\bar{X}, \bar{A}(\cdot), \bar{p}, \bar{c})\) with discount factor \(\beta = (K - 1)/K\) if and only if that sequence is generated by average-cost policy iteration for the MDP \((\bar{X}, A(\cdot), p, c)\).

Idea:

- Write evaluation and improvement steps for \((\bar{X}, \bar{A}(\cdot), \bar{p}, \bar{c})\) in terms of \(\xi\) and the original transition probabilities & costs.
- Use the uniqueness of the solutions obtained in the evaluation step for policy iterations under both criteria.
Complexity estimates: Average-cost policy iterations

**Theorem**

If Assumption HT holds, then for average costs Howard’s policy iteration needs

\[ O(m \cdot K \log K) \]

iterations, and simple policy iteration needs

\[ O(nm \cdot K \log K) \]

iterations.

Theorem follows from the Lemma and Scherrer’s (2013) iteration bounds for Howard’s and simple policy iteration.
Complexity estimates: Average-cost policy iterations

Assumption D

There’s a positive integer $N$ & constant $\alpha$ where

$$\mathbb{P}_x^\phi \{x_N = \ell\} \geq \alpha > 0 \quad \forall x \in X, \phi \in F.$$ 

Corollary

If Assumption D holds, then for average costs Howard’s policy iteration needs

$$O(m \cdot (N/\alpha) \log(N/\alpha))$$

iterations, and simple policy iteration needs

$$O(nm \cdot (N/\alpha) \log(N/\alpha))$$

iterations.

For $N = 1$, Corollary was proved by Feinberg and H. (2013).
Existence of stationary optimal policies: Infinite $\mathbb{X}$

Bounded one-step costs $c \not\Rightarrow$ average-cost optimal policy exists when state space $\mathbb{X}$ is countably infinite.

- Ross (1970)

**Theorem**

*If $c$ is bounded, and Assumption HT holds, then there’s a stationary average-cost optimal policy.*

- Theorem follows from Akian and Gaubert’s (2013) reduction.
- Theorem was proved by Federgruen and Tijms (1978) using a different method.
- Theorem follows from a much more general result covering uncountable state spaces, noncompact action sets, and possibly no special state $\ell$, proved by Feinberg, Kasyanov, and Zadoianchuk (2012).
Existence of stationary optimal policies: Infinite $\mathbb{X}$

**Idea:** Obtain a *bounded* solution $(g, h)$ to the average-cost optimality equation

$$
g + h(x) = \min_{A(x)} \left[ c(x, a) + \sum_{y \in \mathbb{X}} p(y|x, a)h(y) \right], \quad x \in \mathbb{X}
$$

(Derman (1966) showed this suffices) by showing that

$$
T_\xi v(x) := \min_{A(x)} \left[ \frac{c(x, a)}{\xi(x)} + \frac{1}{\xi(x)} \sum_{y \in \mathbb{X}} p(y|x, a)\xi(y)(v(y) - v(\ell)) + \frac{\xi(x) - 1}{\xi(x)} v(\ell) \right]
$$

is a contraction mapping on the space of bounded functions on $\mathbb{X}$. 
Akian and Gaubert (2013) proposed a new reduction of mean-payoff games to discounted games. For MDPs, the complexity results it implies can be proved without non-linear Perron-Frobenius theory. It can also be used to verify the existence of stationary optimal policies.