AMS501: Differential Equations and Boundary Value Problems I
Lecture 17: Local Behavior Near Ordinary Points

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Outline

1. Solution Near an Ordinary Point
2. Translated Series Solutions
3. Types of Recurrence Relations
4. The Legendre Equation
Solution Near an Ordinary Point

Theorem

Suppose that $a$ is an ordinary point of equation

\[ y'' + P(x)y' + Q(x)y = 0, \]

then the equation has two linearly independent solutions, each of the form

\[ y(x) = \sum_{n=0}^{\infty} c_n(x - a)^n. \]

The radius of convergence of any such series solution is at least as large as the distance from $a$ to the nearest (real or complex) singular point of the ODE.
Series solutions can be used to solve initial value problems.

**Example**

Find the general solution in powers of $x$ of

$$(x^2 - 4)y'' + 3xy' + y = 0.$$ 

Then find the particular solution with $y(0) = 4$, $y'(0) = 1.$
Solution of Example 1

**Solution:** The only singular points of the equation is $x = \pm 2$, so the series will have radius of convergence at least 2.

We make the usual substitution

$$y = \sum_{n=0}^{\infty} c_n x^n, \quad y' = \sum_{n=1}^{\infty} n c_n x^{n-1}, \quad y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2},$$

and obtain

$$\sum_{n=2}^{\infty} n(n-1) c_n x^n - 4 \underbrace{\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}}_{= \sum_{n=0}^{\infty} n(n-1) c_n x^n} + 3 \underbrace{\sum_{n=1}^{\infty} n c_n x^n + \sum_{n=0}^{\infty} c_n x^n}_{\sum_{n=0}^{\infty} n c_n x^n} = 0.$$
Solution of Example 1 Cont’d

After shifting the index of summation, we obtain

\[ \sum_{n=0}^{\infty} \left[ (n^2 + 2n + 1)c_n - 4(n + 2)(n + 1)c_{n+2} \right] x^n = 0. \]

The identity principle yields

\[ (n + 1)^2 c_n - 4(n + 2)(n + 1)c_{n+2} = 0, \]

which leads to the recurrence relation

\[ c_{n+2} = \frac{n + 1}{4(n + 2)} c_n \]

for \( n \geq 0 \).
With \( n = 0, 2, \) and \( 4 \) in turn, we get

\[
c_2 = \frac{c_0}{4 \cdot 2}, \quad c_4 = \frac{3c_2}{4 \cdot 4} = \frac{3c_0}{4^2 \cdot 2 \cdot 4}, \quad \text{and} \quad c_6 = \frac{5c_4}{4 \cdot 6} = \frac{3 \cdot 5c_0}{4^3 \cdot 2 \cdot 4 \cdot 6}.
\]

Continuing in this fashion, we evidently would find that

\[
c_{2n} = \frac{3 \cdot 5 \cdots (2n - 1)}{4^n \cdot 2 \cdot 4 \cdot 6 \cdots (2n)} c_0 = \frac{3 \cdot 5 \cdots (2n - 1)}{2^{3n} \cdot 1 \cdot 2 \cdot 3 \cdots n} c_0.
\]

Use the notation of double factorial

\[
(2n + 1)!! = 1 \cdot 3 \cdot 5 \cdots (2n + 1) = \frac{(2n + 1)!}{2^n n!},
\]

we obtain

\[
c_{2n} = \frac{(2n - 1)!!}{2^{3m} \cdot n!} c_0.
\]
Solution of Example 1 Cont’d

With \( n = 1, 3, \) and \( 5, \) we get

\[
\begin{align*}
c_3 &= \frac{2c_1}{4 \cdot 3}, \\
c_5 &= \frac{4c_3}{4 \cdot 5} = \frac{2 \cdot 4c_1}{4^2 \cdot 3 \cdot 5}, \quad \text{and} \\
c_7 &= \frac{6c_5}{4^2 \cdot 7} = \frac{2 \cdot 4 \cdot 6c_1}{4^3 \cdot 3 \cdot 5 \cdot 7}.
\end{align*}
\]

It is apparent that the pattern is

\[
c_{2n+1} = \frac{2 \cdot 4 \cdot 6 \cdots (2n)}{4^n \cdot 3 \cdot 5 \cdots (2n+1)} c_1 = \frac{n!}{2^n \cdot (2n+1)!!} c_1.
\]

The general solution is therefore

\[
y(x) = c_0 \left(1 + \sum_{n=1}^{\infty} \frac{(2n-1)!!}{2^{3m} \cdot n!} x^{2n}\right) + c_1 \left(x + \sum_{n=1}^{\infty} \frac{n!}{2^n \cdot (2n+1)!!} x^{2n+1}\right)
\]

\[
= c_0 \left(1 + \frac{1}{8} x^2 + \frac{3}{128} x^4 + \frac{5}{1024} x^6 + \cdots \right) + c_1 \left(x + \frac{1}{6} x^3 + \frac{1}{30} x^5 + \frac{1}{140} x^7 + \cdots \right)
\]
Solution of Example 1 Cont’d

Because \( y(0) = c_0 \) and \( y'(0) = c_1 \), the given initial conditions imply that \( c_0 = 4 \) and \( c_1 = 1 \). Therefore, the first few terms of the particular solution are

\[
y(x) = 4 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{3}{32}x^4 + \frac{1}{30}x^5 + \cdots
\]

Note that in the above, \( y_0(x) = 1 + \sum_{n=1}^{\infty} \frac{(2n-1)!!}{2^{3m}.n!} x^{2n} \) is the particular solution for the DE with initial-value conditions \( y_0(0) = 1 \) and \( y_0'(0) = 0 \), and \( y_1(x) = x + \sum_{n=1}^{\infty} \frac{n!}{2^n.(2n+1)!!} x^{2n+1} \) is the particular solution for the DE with initial-value conditions \( y_1(0) = 0 \) and \( y_1'(0) = 1 \). \( y_0 \) and \( y_1 \) are linearly independent, so the general solution can be expressed as \( y = c_0y_0 + c_1y_1 \).
Outline

1 Solution Near an Ordinary Point

2 Translated Series Solutions

3 Types of Recurrence Relations

4 The Legendre Equation
If we need to seek a particular solution with given initial values \( y(a) \) and \( y'(a) \), we would have needed the general solution in the form

\[
y(x) = \sum_{n=0}^{\infty} c_n (x - a)^n,
\]

that is, in power of \((x - a)\) rather than in power of \(x\).

The initial conditions \( y(a) = c_0 \) and \( y'(a) = c_1 \) determine the arbitrary constants \( c_0 \) and \( c_1 \).

**Example**

Solve the initial value problem

\[
(t^2 - 2t - 3)y'' + 3(t - 1)y' + y = 0; \quad y(1) = 4, \quad y'(1) = 1.
\]
Solution of Example 2

Solution: We need a general solution of form

\[ y(t) = \sum_{n=0}^{\infty} c_n (t - 1)^n. \]

It simplifies the computation to make the substitution \( x = t - 1 \), so that the series form would be \( \sum_{n=0}^{\infty} c_n x^n \).

Since \( t^2 - 2t - 3 = (t - 1)^2 - 4 \), so the DE becomes

\[ (x^2 - 4)y'' + 3xy' + 4 = 0, \]

with initial conditions \( y(x = 0) = 4 \) and \( y'(x = 0) = 1 \).

The general solution of this DE is

\[ y(x) = c_0 \left( 1 + \frac{1}{8}x^2 + \frac{3}{128}x^4 + \frac{5}{1024}x^6 + \cdots \right) \\
+ c_1 \left( x + \frac{1}{6}x^3 + \frac{1}{30}x^5 + \frac{1}{140}x^7 + \cdots \right), \]
Solution of Example 2 Cont’d

The desired particular solution is

\[ y(t) = 4 \left( 1 + \frac{1}{8}(t - 1)^2 + \frac{3}{128}(t - 1)^4 + \frac{5}{1024}(t - 1)^6 + \cdots \right) \]

\[ + 1 \left( (t - 1) + \frac{1}{6}(t - 1)^3 + \frac{1}{30}(t - 1)^5 + \frac{1}{140}(t - 1)^7 + \cdots \right) \]

\[ = 4 + (t - 1) + \frac{1}{2}(t - 1)^2 + \frac{1}{6}(t - 1)^3 + \frac{3}{32}(t - 1)^4 + \frac{1}{30}(t - 1)^5 + \cdots \]

This series converges if \(-1 < t < 3\). A series such as this can be used to estimate numerical values of the solution. For example,

\[ y(0.8) = 4 + (-0.2) + \frac{1}{2}(-0.2)^2 + \frac{1}{6}(-0.2)^3 + \frac{3}{32}(-0.2)^4 + \cdots \]

\[ \approx 3.8188 \]
Outline

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4. The Legendre Equation
The formula $c_{n+2} = \frac{n+1}{4(n+2)} c_n$ has **two-term** recurrence relations: it expresses each coefficient in terms of one of the preceding coefficients.

A **many-term** recurrence relation expresses each coefficient in the series in terms of two or more preceding coefficients.

Many-term recurrence may be inconvenient or even impossible to find a formula for $c_n$ in terms of $n$, but three-term recurrence relationship is sometimes manageable.

**Example**

Find two linearly independent solutions of

$$y'' - xy' - x^2 y = 0.$$
Solution of Example 3

Solution: We make the usual substitution of the power series and obtain the equation

\[
\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} - \sum_{n=1}^{\infty} nc_n x^n - \sum_{n=0}^{\infty} c_n x^{n+2} = 0.
\]

\[
= 2c_2 + 6c_3 x + \sum_{n=2}^{\infty} (n+2)(n+1)c_{n+2} x^n = c_1 x + \sum_{n=2}^{\infty} nc_n x^n = \sum_{n=2}^{\infty} c_{n-2} x^n
\]

\[
2c_2 + 6c_3 x - c_1 x + \sum_{n=2}^{\infty} [(n+2)(n+1)c_{n+2} - nc_n - c_{n-2}] x^n = 0.
\]

Therefore,

\[
c_2 = 0,
\]

\[
c_3 = \frac{1}{6} c_1
\]

\[
c_{n+2} = \frac{nc_n + c_{n-2}}{(n+2)(n+1)}
\]

for \( n \geq 2 \).
Solution of Example 3 Cont’d

In particular,

\[ c_4 = \frac{2c_2 + c_0}{12}, \quad c_5 = \frac{3c_3 + c_1}{20}, \quad c_6 = \frac{4c_4 + c_2}{30}, \]
\[ c_7 = \frac{5c_5 + c_3}{42}, \quad c_8 = \frac{6c_6 + c_4}{56}. \]

Since \( c_2 = 0 \) and \( c_3 = \frac{1}{6} c_1 \), all the coefficients are given in terms of arbitrary constants \( c_0 \) and \( c_1 \).

To obtain first solution \( y_1 \), let \( c_0 = 1 \) and \( c_1 = 0 \), so that \( c_2 = c_3 = 0 \). Then

\[ c_4 = \frac{1}{12}, \quad c_5 = 0, \quad c_6 = \frac{1}{90}, \quad c_7 = 0, \quad c_8 = \frac{3}{1120} \]

and thus

\[ y_1(x) = 1 + \frac{1}{12}x^4 + \frac{1}{90}x^6 + \frac{3}{1120}x^8 + \cdots \]

which contains only even degree terms because \( c_1 = c_3 = 0 \).
To obtain second solution $y_2$, let $c_0 = 0$ and $c_1 = 1$, so that $c_2 = 0$ and $c_3 = 1/6$. Then,

$$c_4 = 0, \quad c_5 = \frac{3}{40}, \quad c_6 = 0, \quad c_7 = \frac{13}{1008}$$

and thus

$$y_2(x) = x + \frac{1}{6} x^3 + \frac{3}{40} x^5 + \frac{13}{1008} x^7 + \cdots$$

which contains only odd degree terms because $c_0 = c_2 = 0$. The solutions $y_1$ and $y_2$ are linearly independent. A general solution is a linear combination of the power series $y_1$ and $y_2$. There is no singular point, so the power series converge for all $x$. ■
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4. The Legendre Equation
The Legendre equation of order $\alpha$ is the second-order linear DE

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0,$$

where $\alpha$ is a real number and $\alpha > -1$.

It has applications in numerical integration formulas and in determining the steady-state temperature in a solid spherical ball with temperature at boundary points are known.

The only singular points are $x = \pm 1$, so it has two linearly independent solutions that can be expressed as power series in powers of $x$ with radius of convergence at least 1.
Solution of Legendre Equation

- The substitution \( y = \sum_{m=0}^{\infty} c_m x^m \) leads to

\[
c_{m+2} = -\frac{(\alpha - m)(\alpha + m + 1)}{(m + 1)(m + 2)} c_m
\]

for \( m \geq 0 \). (Here we use \( m \) as the index of summation because \( n \) will play another role.)

- In terms of arbitrary constants \( c_0 \) and \( c_1 \), we have

\[
c_2 = -\frac{\alpha(\alpha + 1)}{2!} c_0
\]

\[
c_3 = -\frac{(\alpha - 1)(\alpha + 2)}{3!} c_1
\]

\[
c_4 = \frac{\alpha(\alpha - 2)(\alpha + 1)(\alpha + 3)}{4!} c_0
\]

\[
c_5 = \frac{(\alpha - 1)(\alpha - 3)(\alpha + 2)(\alpha + 4)}{5!} c_1.
\]
Solution of Legendre Equation Cont’d

By induction, it can be shown that

\[ c_{2m} = (-1)^m \frac{\alpha(\alpha - 2)(\alpha - 4) \cdots (\alpha - 2m + 2)(\alpha + 1)(\alpha + 3) \cdots (\alpha + 2m - 1)}{(2m)!} c_0 \]

and

\[ c_{2m+1} = (-1)^m \frac{(\alpha - 1)(\alpha - 3) \cdots (\alpha - 2m + 1)(\alpha + 2)(\alpha + 4) \cdots (\alpha + 2m)}{(2m + 1)!} c_0. \]

Therefore, we get two linearly independent power series solutions

\[ y_1 = c_0 \sum_{m=0}^{\infty} (-1)^m a_{2m} x^{2m} \quad \text{and} \quad y_2 = c_1 \sum_{m=0}^{\infty} (-1)^m a_{2m+1} x^{2m+1} \]

of Legendre’s equation of order \( \alpha \).
Suppose $\alpha = n$ is a nonnegative integer.

- If $\alpha = n$ is even, then $a_{2m} = 0$ when $2m > n$. So $y_1$ is a polynomial of degree $n$, and $y_2$ is an infinite series.
- If $\alpha = n$ is odd, then $a_{2m+1} = 0$ when $2m + 1 > n$. In this case, $y_2$ is polynomial of degree $n$ and $y_1$ is an infinite series.

By properly choosing $c_0$ (if $n$ is even) and $c_1$ (if $n$ is odd), we get a sequence of $n$th degree polynomials called Legendre polynomials

$$P_n(x) = \sum_{k=0}^{[n/2]} \frac{(-1)^k(2n - 2k)!}{2^n k!(n-k)!(n-2k)!} x^{n-2k}.$$
Legendre Polynomials

- First six Legendre polynomials are

\[ P_0(x) \equiv 1, \quad P_1(x) = x \]
\[ P_2(x) = \frac{1}{2}(3x^2 - 1), \quad P_3(x) = \frac{1}{2}(5x^3 - 3x), \]
\[ P_4(x) = \frac{1}{8}(35x^4 - 30x^2 + 3), \quad P_5(x) = \frac{1}{8}(63x^5 - 70x^3 + 15x). \]

- Legendre polynomials are orthogonal to each other over \([-1, 1]\), in that

\[
\int_{-1}^{1} P_i(x)P_j(x) \, dx = 0
\]

if \( i \neq j \).