# AMS526: Numerical Analysis I (Numerical Linear Algebra for Computational and Data Sciences)

## Lecture 17: Other Eigenvalue Algorithms; Generalized Eigenvalue Problems

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# Outline

# Three Alternative Algorithms

- Jacobi algorithm: earliest known method
- Bisection method: standard way for finding few eigenvalues
- Divide-and-conquer: faster than QR and amenable to parallelization

# The Jacobi Algorithm

- Diagonalize $2 \times 2$ real symmetric matrix by *Jacobi rotation*

$$J^T \begin{bmatrix} a & d \\ d & b \end{bmatrix} J = \begin{bmatrix} \neq 0 & 0 \\ 0 & \neq 0 \end{bmatrix}$$

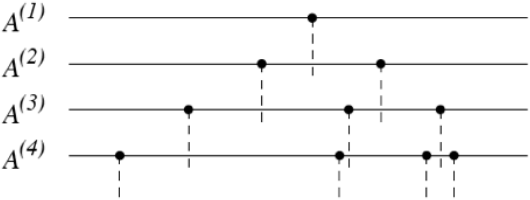
*where* $J = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$, and $\tan(2\theta) = 2d/(b-a)$

- What are its similarity and differences with Givens rotation?
- Iteratively apply transformation to two rows and two corresponding columns of $A \in \mathbb{R}^{n \times n}$
- Need not tridiagonalize first, but loop over all pairs of rows and columns by choosing greedily or cyclically
- Magnitude of nonzeros shrink steadily, converging quadratically
- In each iteration, $O(n^2)$ Jacobi rotation, $O(n)$ operations per rotation, leading to $O(n^3 \log(|\log \epsilon_{\text{machine}})|))$ flops total
- Jacobi method is easy to parallelize (QR algorithm does not scale well), delivers better accuracy than QR algorithm, but far slower than QR algorithm

# Method of Bisection

- Idea: Search the real line for roots of $p(x) = \det(A - xI)$
- Finding roots from coefficients is highly unstable, but computing $p(x)$ from given $x$ is stable (e.g., can be computed using Gaussian elimination with partial pivoting)
- Let $A^{(i)}$ denote principal square submatrix of dimension $i$ for irreducible matrix $A$ (note: different from notation in QR algorithm)
- Key property: eigenvalues of $A^{(1)}, \ldots, A^{(n)}$ strictly interlace

$$\lambda_j^{(k+1)} < \lambda_j^{(k)} < \lambda_{j+1}^{(k+1)}$$



# Method of Bisection

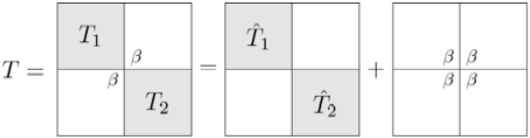
- Interlacing property allows us to determine number of negative eigenvalues of $A$, which is equal to number of sign changes in *Sturm* sequence
$$1, \det(A^{(1)}), \det(A^{(2)}), \ldots, \det(A^{(n)})$$
- Shift $A$ to get number of eigenvalues in $(-\infty, b)$ and $(-\infty, a)$, and in turn $[a, b)$
- Three-term recurrence for determinants for tridiagonal matrices
$$\det(A^{(k)}) = a_{k,k}\det(A^{(k-1)}) - a_{k,k-1}^2\det(A^{(k-2)})$$
- With shift $xI$ and $p^{(k)}(x) = \det(A^{(k)} - xI)$:
$$p^{(k)}(x) = (a_{k,k} - x)p^{(k-1)}(x) - a_{k,k-1}^2 p^{(k-2)}(x)$$
- Bisection algorithm can locate eigenvalues in arbitrarily small intervals
- $O(n|\log(\epsilon_{\text{machine}})|)$ flops per eigenvalue, always high relative accuracy

# Notes on Bisection

- It is standard algorithm if one needs a few eigenvalues
- Key step of bisection is to determine the inertia (i.e., the numbers of positive, negative, and zero eigenvalues) of $A - \mu I$
- *Sylvester's Law of Inertia*: inertia is invariant under *congruence transformation* $SAS^T$, where $S$ is nonsingular (proved in 1852)
- Therefore, $LDL^T$ may be used to determine inertia

# Divide-and-Conquer Algorithm

- Split symmetric algorithm $T$ into submatrices




- Sum of $2 \times 2$ block-diagonal matrix and rank-one correction
- Split $T$ in equal sizes and compute eigenvalues of $\hat{T}_1$ and $\hat{T}_2$ recursively
- Solve a nonlinear problem to get eigenvalues of $T$ from those of $\hat{T}_1$ and $\hat{T}_2$

# Divide-and-Conquer Algorithm

- Suppose diagonalizations $\hat{T}_1 = Q_1 D_1 Q_1^T$ and $\hat{T}_2 = Q_2 D_2 Q_2^T$ have been computed. We then have

$$T = \begin{bmatrix} Q_1 & \\ & Q_2 \end{bmatrix} \left( \begin{bmatrix} D_1 & \\ & D_2 \end{bmatrix} + \beta z z^T \right) \begin{bmatrix} Q_1^T & \\ & Q_2^T \end{bmatrix}$$

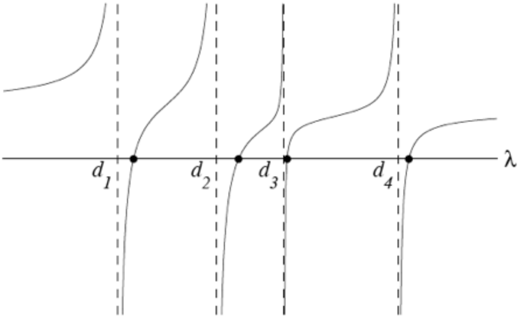
  with $z^T = (q_1^T, q_2^T)$, where $q_1^T$ is last row of $Q_1$ and $q_2^T$ is first row of $Q_2$

- This is similarity transformation: Find eigenvalues of diagonal matrix plus rank-one correction

# Divide-and-Conquer Algorithm

- Eigenvalues of $D + ww^T$ are the roots of rational function

$$f(\lambda) = 1 + \sum_{j=1}^{n} \frac{w_j^2}{d_j - \lambda}$$



# Divide-and-Conquer Algorithm

- Solve *secular equation* $f(\lambda) = 0$ with quadratic convergence
- $O(n \log|\log(\epsilon_{\text{machine}})|)$ flops per root; $O(n^2 \log|\log(\epsilon_{\text{machine}})|)$ flops for all roots
- Total cost for divide-and-conquer algorithm is

$$O\left(\sum_{k=1}^{\log n} 2^{k-1}\left(\frac{n}{2^{k-1}}\right)^2\right) = O(n^2),$$

  where constant depends on $\log|\log(\epsilon_{\text{machine}})|$
- For computing eigenvalues only, most of operations are spent in tridiagonal reduction, and constant in "Phase 2" is not important
- However, for computing eigenvectors, divide-and-conquer reduces phase 2 to $4n^3/3$ flops compared to $6n^3$ for QR

# Outline

# Generalized Eigenvalue Problem

- Generalized eigenvalue problem has the form

$$Ax = \lambda Bx,$$

  where $A$ and $B$ are $n \times n$ matrices
- For example, in structural vibration problems, $A$ represents the *stiffness matrix*, $B$ the *mass matrix*, and eigenvalues and eigenvectors determine natural frequencies and modes of vibration of structures
- If $A$ or $B$ is nonsingular, then it can be converted into standard eigenvalue problem $(B^{-1}A)x = \lambda x$ or $(A^{-1}B)x = (1/\lambda)x$
- If $A$ and $B$ are both symmetric, preceding transformation loses symmetry and in turn may lose orthogonality of generalized eigenvectors. If $B$ is positive definite, alternative transformation is

$$(L^{-1}AL^{-T})y = \lambda y, \text{ where } B = LL^T \text{ and } y = L^T x$$

- If $A$ and $B$ are both singular or indefinite, then use *QZ algorithm* to reduce $A$ and $B$ into triangular matrices simultaneously by orthogonal transformation (see Golub and van Loan for detail)