AMS 526 Sample Questions for Final Exam

December 15, 2008

Note: These sample questions focus on materials after Test 2. Refer to previous sample and test questions for earlier materials.

1. Answer true or false with a brief justification. (No credit without justification.)

   (a) If $\boldsymbol{A} \in \mathbb{C}^{m\times m}$ is symmetric positive definite, then for any *nonsingular* $\boldsymbol{B} \in \mathbb{C}^{m\times m}$ the eigenvalues of $\boldsymbol{B}\boldsymbol{A}\boldsymbol{B}^T$ are all real and positive. (Hint: Use properties of positive definiteness.)

   (b) The eigenvalues of a real matrix are not necessarily real but their product must be real. (Hint: Product of eigenvalues is the determinant.)

   (c) If a matrix is nonhermitian, then it is not unitarily diagonalizable. (Hint: A matrix is unitarily diagonalizable if and only if it is normal.)

   (d) Reduction to Hessenberg form by Householder reflectors preserves singular values and eigenvalues.

   (e) If $\boldsymbol{A}$ is unitary, then it has a full set of orthonormal eigenvectors. (Hint: Unitary matrices are normal.)

   (f) The conjugate gradient method is appropriate to solving a linear system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ if $\boldsymbol{A}$ is symmetric and sparse.

   (g) The conjugate gradient method for solving linear system $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ is optimal in the sense that the error in $\boldsymbol{x}$ is minimized in 2-norm within the Krylov subspace at each step.

   (h) If a matrix is both triangular and normal, then it must be diagonal.

2. Suppose $\boldsymbol{A}$ is symmetric positive definite and $\boldsymbol{B}$ is symmetric matrix. Let $\boldsymbol{A} = \boldsymbol{R}^T\boldsymbol{R}$ be the Cholesky factorization of $\boldsymbol{A}$. Show that $\boldsymbol{A}\boldsymbol{B}$, $\boldsymbol{B}\boldsymbol{A}$, and $\boldsymbol{R}\boldsymbol{B}\boldsymbol{R}^T$ have the same set of eigenvalues. (Hint: Construct similarity transformations.)

3. Prove or disprove. Let $\boldsymbol{A} \in \mathbb{C}^{m\times m}$ be a nonhermitian matrix and $\boldsymbol{q} \in \mathbb{C}^m$ be a vector with $\|\boldsymbol{q}\|_2 = 1$.

   (a) There exists a unitary matrix $\boldsymbol{Q} \in \mathbb{C}^{m\times m}$ whose **first** column is $\boldsymbol{q}$, such that $\boldsymbol{Q}^*\boldsymbol{A}\boldsymbol{Q}$ is an upper-Hessenberg matrix. (Hint: Consider Arnoldi iterations.)

   (b) There exists a unitary matrix $\boldsymbol{Q} \in \mathbb{C}^{m\times m}$ whose **last** column is $\boldsymbol{q}$, such that $\boldsymbol{Q}^*\boldsymbol{A}\boldsymbol{Q}$ is a lower-Hessenberg matrix. (Hint: Consider introducing a "flipped" permutation matrix.)

4. Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{C}^m$. Show that $\det(\boldsymbol{I} - \boldsymbol{x}\boldsymbol{y}^*) = 1 - \boldsymbol{y}^*\boldsymbol{x}$. (Hint: Show that $1 - \boldsymbol{y}^*\boldsymbol{x}$ is an eigenvalue and 1 is another eigenvalue with multiplicity $m - 1$.)

5. A matrix $\boldsymbol{A}$ is said to be *nilpotent* if $\boldsymbol{A}^k = \boldsymbol{0}$ for some positive integer k.

   (a) Show that if $\boldsymbol{A}$ is nilpotent, then all of the eigenvalues of $\boldsymbol{A}$ are zero.

   (b) Show that if $\boldsymbol{A}$ is both nilpotent and normal (i.e., $\boldsymbol{A}^*\boldsymbol{A} = \boldsymbol{A}\boldsymbol{A}^*$), then $\boldsymbol{A} = \boldsymbol{0}$.

6. Let $\boldsymbol{A}$ be skew hermitian, i.e., $\boldsymbol{A}^* = -\boldsymbol{A}$.

   (a) Prove that the eigenvalues of $\boldsymbol{A}$ are purely imaginary.

   (b) Show that its eigenvectors corresponding to different eigenvalues are orthogonal to each other.

(c) Prove that $\boldsymbol{I} - \boldsymbol{A}$ is nonsingular.

7. What methods would you choose in solving the following problems? Briefly explain your reasons.

   (a) Determine the eigenvalues within a certain interval.

   (b) Determine the largest eigenvalue.

   (c) Determine the smallest eigenvalue.

   (d) Find extreme eigenvalues.

   (e) Given an approximate eigenvalue λ, obtain an approximate eigenvector.

   (f) Given an approximate eigenvector $\boldsymbol{x}$, obtain an good approximate eigenvalue.