AMS 526 Homework 2, Fall 2008

Due: Tuesday 10/07 in class

1. (10 points) Let $\boldsymbol{P} \in \mathbb{C}^{m\times m}$ be an orthogonal projector. Prove that $\boldsymbol{I} - 2\boldsymbol{P}$ is unitary.

2. (20 points) Let $\boldsymbol{P} \in \mathbb{C}^{m\times m}$ be a nonzero projector. Show that $\|\boldsymbol{P}\|_2 \geq 1$, where the equality holds if and only if $\boldsymbol{P} = \boldsymbol{P}^*$.

3. (20 points) Let $\boldsymbol{A}$ be an $m \times n$ matrix with $m \geq n$, and let $\boldsymbol{A} = \hat{\boldsymbol{Q}}\hat{\boldsymbol{R}}$ be a reduced QR factorization. Suppose $\hat{\boldsymbol{R}}$ has k nonzero diagonal entries for some k with $0 \leq k < n$. Is the rank of $\boldsymbol{A}$ equal to k, greater than k, or less than k? Justify your answer by giving a proof.

4. (50 points) Implement both the classical and modified Gram-Schmidt procedures in the C programming language. Use each to generate an orthogonal matrix $\boldsymbol{Q}$ whose columns form an orthonormal basis for the column space of the $n \times n$ Hilbert matrix $\boldsymbol{H}$, for $n = 2, \ldots, 12$. The Hilbert matrix has entries $h_{ij} = 1/(i + j - 1)$. For example, a 3×3 Hilbert matrix has entries

$$\begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix}.$$

   As a measure of the quality of the results (specifically, the potential loss of orthogonality), plot the quantity $-\log_{10}(\|\boldsymbol{I} - \boldsymbol{Q}^T\boldsymbol{Q}\|_F)$, which can be interpreted as "digits of accuracy," for each method as a function of n. In addition, try applying the classical procedure twice (i.e., apply your classical Gram-Schmidt routine to its own output $\boldsymbol{Q}$ to obtain a new $\boldsymbol{Q}$), and plot the resulting departure from orthogonality. How do the three methods compare in speed, storage, and accuracy?
   Implement your code by following the template files given on the class website. Submit your C code, the plots, and your conclusions of the comparative study.