AMS 526 Homework 4, Fall 2008

Due: Thursday 11/06 in class.

1. (20 points) Exercise 18.1 of textbook.

2. (20 points) Exercise 21.4 of textbook.

3. (20 points) Exercise 21.6 of textbook.

4. (40 points) Write a routine to estimate the condition number of a real matrix $\boldsymbol{A}$ using 1-norm. You will need to compute $\|\boldsymbol{A}\|_1$, which is easy, and estimate $\|\boldsymbol{A}^{-1}\|_1$, which is more challenging. One way to estimate $\|\boldsymbol{A}^{-1}\|_1$ is to take it as the ratio $\|\boldsymbol{z}\|_1/\|\boldsymbol{y}\|_1$, where $\boldsymbol{z}$ is the solution to $\boldsymbol{A}\boldsymbol{z} = \boldsymbol{y}$ and $\boldsymbol{y}$ is picked by some heuristic to maximize the ratio. Specifically, we choose $\boldsymbol{y}$ as the solution to the system $\boldsymbol{A}^T\boldsymbol{y} = \boldsymbol{c}$ for some vector $\boldsymbol{c}$ whose components are all ± 1. We determine the sign of the components in $\boldsymbol{c}$ by utilizing the LU factorization with pivoting $\boldsymbol{P}\boldsymbol{A} = \boldsymbol{L}\boldsymbol{U}$. Note that if $\boldsymbol{c}$ were known, then $\boldsymbol{A}^T\boldsymbol{y} = \boldsymbol{c}$ would be solved by $\boldsymbol{U}^T\boldsymbol{v} = \boldsymbol{c}$ (forward substitution) and $\boldsymbol{L}^T\boldsymbol{P}\boldsymbol{y} = \boldsymbol{v}$ (back substitution). We modify the forward substitution step so that at its ith step (i.e., the step of computing v_i), we choose c_i to be either 1 or -1, depending on which choice makes $|v_i|$ larger. You need to write a modified forward substitution routine to accomplish this. After obtaining v, solve $\boldsymbol{L}^T\boldsymbol{P}\boldsymbol{y} = \boldsymbol{v}$ in the usual way to obtain $\boldsymbol{y}$. The idea here is that any ill-conditioning in $\boldsymbol{A}$ will be reflected in $\boldsymbol{U}$, resulting in a relatively large $\boldsymbol{v}$. The relative well-conditioning unit triangular matrix $\boldsymbol{L}$ will then preserve this relationship, resulting in relatively large $\boldsymbol{y}$.

   Test your program on the Hilbert matrix of order $n = 2, 3, \ldots, 12$. To check the quality of your estimates, compute $\boldsymbol{A}^{-1}$ explicitly using the LU factorization that was already computed and then compute the condition number $\|\boldsymbol{A}\|_1\|\boldsymbol{A}^{-1}\|_1$. Plot the estimated condition numbers and the explicitly computed condition numbers for the Hilbert matrix of order $n = 2, 3, \ldots, 12$ (using the horizontal axis for n and the vertical axis for the condition numbers). Compare the required flops of the two approaches (i.e., estimation and explicit computation), and also plot their running times for different n. To obtain reliable timing of a procedure, you may need to run it repeatedly for hundreds of iterations and then take the average.

   Submit your programs, the plots, and a brief discussion of your results. For its complexity, you are recommended to use MATLAB or Octave for this assignment.