

# AMS526: Numerical Analysis I (Numerical Linear Algebra)

## Lecture 1: Course Overview & Matrix-Vector Multiplication

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# Outline

1 Course Overview

2 Matrix-Vector Multiplication

# Course Description

- What is numerical linear algebra?
  - ▶ Solving linear algebra problems using efficient algorithms on computers
- Topics: Direct and iterative methods for solving simultaneous linear equations. Matrix factorization and decomposition. Conditioning, stability, and efficiency. Computation of eigenvalues and eigenvectors
- Prerequisite/Co-requisite:
  - ▶ AMS 505 (Applied Linear Algebra)
  - ▶ Programming in C (AMS 595, Fundamentals of Computing, is highly recommended)
- Required Textbook: *Numerical Linear Algebra*, by Lloyd N. Trefethen and David Bau, III, SIAM, 1997, ISBN 0-89871-361-7.
- Reference Book (not required): *Matrix Computations*, 3rd edition, by Gene H. Golub and Charles F. Van Loan, John Hopkins University Press, 1996, ISBN 0-8018-5414-8.

# Why Learn Numerical Linear Algebra?

- Numerical linear algebra is foundation of scientific computations
- Many problems ultimately reduce to linear algebra concepts or algorithms, either analytical or computational
- Examples: Finite-element analysis, data fitting, PageRank (Google)

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## The \$25,000,000,000 Eigenvector: The Linear Algebra behind Google\*

Kurt Bryan<sup>†</sup>  
Tanya Leise<sup>‡</sup>

**Abstract.** Google's success derives in large part from its PageRank algorithm, which ranks the importance of web pages according to an eigenvector of a weighted link matrix. Analysis of the PageRank formula provides a wonderful applied topic for a linear algebra course. Instructors may assign this article as a project to more advanced students or spend one or two lectures presenting the material with assigned homework from the exercises. This material also complements the discussion of Markov chains in matrix algebra. Maple and *Mathematica* files supporting this material can be found at [www.rose-hulman.edu/~bryan](http://www.rose-hulman.edu/~bryan).

- Focus: Fundamental concepts, algorithms, and programming

## Course Outline

- Matrix fundamentals, multiplications, orthogonality, norms, and SVD (2.5 weeks).
- QR factorization, projectors, Gram-Schmidt algorithm, Householder triangulation (2 weeks).
- Conditioning and stability (1.5 weeks).
- Solution of linear system of equations, Gaussian elimination, pivoting, LU factorization, Cholesky factorization (2 weeks).
- Eigenvalue problems, Hessenberg tridiagonalization, Rayleigh quotient, power method, inverse power method and shifting (2.5 weeks).
- Iterative methods, Lanczos iteration (1 weeks).
- Linear algebra software packages (1 weeks).
- Course webpage:  
[http://www.ams.sunysb.edu/~jiao/teaching/ams526\\_fall108](http://www.ams.sunysb.edu/~jiao/teaching/ams526_fall108)

Note: Course schedule online is tentative and is subject to change.

# Course Policy

- Assignments (written or programming)
  - ▶ Assignments are due in class one to two weeks after assigned
  - ▶ You can discuss course materials and homework problems with others, but you must write your answers completely independently
  - ▶ Do NOT copy solutions from any source. Do NOT share your solutions to others
- Exams and tests
  - ▶ All exams are closed-book
  - ▶ However, one-page cheat sheet is allowed
- Attendance
  - ▶ You are expected to attend all the lectures and exams
  - ▶ Lecture notes are sketchy so reading them alone is not enough
- Grading
  - ▶ Assignments: 30%
  - ▶ Two tests: 40%
  - ▶ Final exam: 30%

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## Definition

- Matrix-vector product  $\mathbf{b} = \mathbf{A}\mathbf{x}$

$$b_i = \sum_{j=1}^n a_{ij}x_j$$

- All entries belong to  $\mathbb{C}$ , the field of complex numbers. The space of  $m$ -vectors is  $\mathbb{C}^n$ , and the space of  $m \times n$  matrices is  $\mathbb{C}^{m \times n}$ .
- The map  $\mathbf{x} \mapsto \mathbf{A}\mathbf{x}$  is *linear*, which means that for any  $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$  and any  $\alpha \in \mathbb{C}$

$$\mathbf{A}(\mathbf{x} + \mathbf{y}) = \mathbf{A}\mathbf{x} + \mathbf{A}\mathbf{y},$$

$$\mathbf{A}(\alpha\mathbf{x}) = \alpha\mathbf{A}\mathbf{x}.$$

- In lecture notes, I use **boldface** UPPERCASE letters (such as  $\mathbf{A}$ ) for matrices and **boldface** lowercase letters (such as  $\mathbf{x}$ ) for vectors.

## Pseudo-Code for Matrix-Vector Product

Pseudo-code for  $\mathbf{b} = \mathbf{Ax}$

**for**  $i = 1$  to  $m$  **do**

$b(i) = 0;$

**for**  $j = 1$  to  $n$  **do**

$b(i) = b(i) + A(i,j) * x(j);$

**end for**

**end for**

Note: Pseudo-code are not real code in that they do not run on any computer. They are human readable, but they should be straightforward to convert into real computer codes in any programming languages.

# Linear Combination

- Alternatively, matrix-vector product can be viewed as

$$\mathbf{b} = \mathbf{Ax} = \sum_{j=1}^n x_j \mathbf{a}_j$$

- Two different views of matrix-vector products:
  - 1  $b_i = \sum_{j=1}^n a_{ij}x_j$ :  $\mathbf{A}$  acts on  $\mathbf{x}$  to produce  $\mathbf{b}$ ; scalar operations;
  - 2  $\mathbf{b} = \sum_{j=1}^n x_j \mathbf{a}_j$ :  $\mathbf{x}$  acts on  $\mathbf{A}$  to produce  $\mathbf{b}$ ; vector operations.

# Matrix-Matrix Multiplication

- If  $\mathbf{A}$  is  $l \times m$  and  $\mathbf{C}$  is  $m \times n$ , then  $\mathbf{B} = \mathbf{AC}$  is  $l \times n$ , with entries defined by

$$b_{ij} = \sum_{k=1}^m a_{ik} c_{kj}.$$

- Written in columns, we have

$$\mathbf{b}_j = \mathbf{A}\mathbf{c}_j = \sum_{k=1}^m c_{kj} \mathbf{a}_k.$$

- In other word, each column of  $\mathbf{B}$  is a linear combination of the columns of  $\mathbf{A}$ .

# Pseudo-Code for Matrix-Matrix Multiplication

Pseudo-code for  $B = AC$

**for**  $i = 1$  to  $l$  **do**

**for**  $j = 1$  to  $n$  **do**

$B(i,j) = 0;$

**for**  $k = 1$  to  $m$  **do**

$B(i,j) = B(i,j) + A(i,k) * C(k,j);$

**end for**

**end for**

**end for**

# Range and Null Space

## Definition

The *range* of a matrix  $\mathbf{A}$ , written as  $\text{range}(\mathbf{A})$ , is the set of vectors that can be expressed as  $\mathbf{Ax}$  for some  $\mathbf{x}$ .

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*range*( $\mathbf{A}$ ) is the space spanned by the columns of  $\mathbf{A}$ .

Therefore, the *range* of  $\mathbf{A}$  is also called the *column space* of  $\mathbf{A}$ .

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## Definition

The *null space* of  $\mathbf{A} \in \mathbb{C}^{m \times n}$ , written as  $\text{null}(\mathbf{A})$ , is the set of vectors  $\mathbf{x}$  that satisfy  $\mathbf{A}\mathbf{x} = \mathbf{0}$ .

Entries of  $\mathbf{x} \in \text{null}(\mathbf{A})$  give coefficient of  $\sum x_i \mathbf{a}_i = \mathbf{0}$ .

Note: The null space of  $\mathbf{A}$  is in general **not** a complementary subspace of  $\text{range}(\mathbf{A})$ .

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## Definition

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**Answer:** No.

- We therefore simply say the *rank* of a matrix.

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**Question:** Given  $\mathbf{A} \in \mathbb{C}^{m \times n}$ , what does  $\dim(\text{null}(\mathbf{A})) + \text{rank}(\mathbf{A})$  equal to?

**Answer:**  $n$ .