

# AMS526: Numerical Analysis I (Numerical Linear Algebra)

Review Session

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## Announcement: Final Exam

- Tuesday **12/23** between **11am** and **1:30pm** in classroom
- Final exam will be **comprehensive** covering **all** the material from the semester
- About 40–50% will be on material after Test 2 (i.e., eigenvalue problems and iterative methods, which are closely connected with earlier materials, especially QR and SVD)
- As usual, you can have a **single-sided, one-page, letter-size** (8.5inx11in) cheat sheet

# Topics Covered in The Course

- Fundamental concepts: norms, orthogonality, conditioning, stability
- Least squares problems using direct method (QR factorization)
- Singular value decomposition, properties, and relationship with eigenvalue problems
- Eigenvalue problems, properties, and algorithms (QR algorithm and Lanczos iterations)
- Solving linear systems using direct (Gaussian elimination) and iterative (Krylov subspace) methods
- Conditioning of problems, stability and backward stability of algorithms
- Efficiency of algorithms, convergence rate of iterative methods

# Matrix Properties and Transformations

- Properties

- ▶ Hermitian (symmetric), skew symmetry, positive definite
- ▶ unitary (orthogonal), normal, (orthogonal and oblique) projection matrix
- ▶ singular/nonsingular, defective/nondefective
- ▶ triangular, Hessenberg, tridiagonal, diagonal, Jordan-form, sparse

- Transformations

- ▶ orthogonalization (Gram-Schmidt)
- ▶ triangularization (Gaussian elimination, Cholesky factorization, Householder QR)
- ▶ reduction to Hessenberg or tridiagonal form
- ▶ similarity transformation and unitary similarity transformation (Schur factorization)
- ▶ congruence transformation (preserves symmetry and inertia)

# Fundamental Algorithms

- QR factorization using classical and modified Gram-Schmidt
- QR factorization using Householder triangularization
- Gaussian elimination with partial pivoting and Cholesky factorization
- Reduction to Hessenberg/tridiagonal form for eigenvalue problems
- QR algorithm with or without shifts for eigenvalue problems
- Lanczos iterations and Conjugate Gradients
- Do not need to memorize the details of the algorithms
- Understand when they work, how they work, why they work, and how well they work
- Understand relationships among each other: how one transforms into another, and to make an intelligent choice

# Eigenvalue Problem

- *Eigenvalue problem* of  $m \times m$  matrix  $\mathbf{A}$  is  $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$
- *Characteristic polynomial* is  $\det(\mathbf{A} - \lambda\mathbf{I})$
- *Eigenvalue decomposition* of  $\mathbf{A}$  is  $\mathbf{A} = \mathbf{X}\mathbf{\Lambda}\mathbf{X}^{-1}$  (does not always exist)
- *Geometric multiplicity* of  $\lambda$  is  $\dim(\text{null}(\mathbf{A} - \lambda\mathbf{I}))$ , and *algebraic multiplicity* of  $\lambda$  is its multiplicity as a root of  $p_{\mathbf{A}}$ , where algebraic multiplicity  $\geq$  geometric multiplicity
- *Similar* matrices have the same eigenvalues, and algebraic and geometric multiplicities
- *Schur factorization*  $\mathbf{A} = \mathbf{Q}\mathbf{T}\mathbf{Q}^*$  uses unitary similarity transformations

# Eigenvalue Algorithms

- Underlying concepts: power iterations, Rayleigh quotient, inverse iterations, convergence rate
- *Schur factorization* is typically done in two steps
  - ▶ Reduction to Hessenberg form for nonhermitian matrices or reduction to tridiagonal form for hermitian matrices by unitary similarity transformation
  - ▶ Finding eigenvalues of Hessenberg or **tridiagonal** form
- Finding eigenvalue of tridiagonal forms
  - ▶ QR algorithm with shifts, and their interpretations as (inverse) simultaneous iterations
  - ▶ Others: Bisection and divide-and-conquer
- Alternative method is Jacobi method for symmetric matrices using Jacobi rotations

# Relationship between SVD and Eigenvalue Decomposition

- SVD works for all matrices (even rectangular matrices), but eigenvalue decomposition (i.e., diagonalization) works only for nondefective square matrices
- Singular vectors are always orthonormal and singular values are always real, while eigenvectors may not be orthogonal and eigenvalues may be complex numbers
- For normal matrices, singular values and eigenvalue are particularly closely related, which make them particularly powerful analytical tools

# Iterative Methods

- Advantages and disadvantages of iterative methods vs. direct methods
- We focus on Krylov subspace methods for symmetric matrices
- Given  $\mathbf{A}$  and  $\mathbf{b}$ , *Krylov subspace* is  $\{\mathbf{b}, \mathbf{A}\mathbf{b}, \mathbf{A}^2\mathbf{b}, \dots, \mathbf{A}^k\mathbf{b}\}$
- Key observation: QR factorization of leading vectors of Krylov subspace leads to Hessenberg form for nonsymmetric matrices and tridiagonal form for symmetric matrices
- Lanczos iterations takes advantage of the tridiagonal form to get three-term recurrence version of Arnoldi iterations
- Conjugate gradient methods for solving SPD linear systems: solution as quadratic optimization problem, finite-termination properties with exact arithmetic, and convergence with floating-point arithmetic