AMS526: Numerical Analysis I
(Numerical Linear Algebra)
Lecture 17: QR Algorithm and Unnormalized Simultaneous Iteration

Xiangmin Jiao

SUNY Stony Brook
QR Algorithm

- Most basic version of QR algorithm is remarkably simple:

  Algorithm: “Pure” QR Algorithm
  \[
  A^{(0)} = A \\
  \text{for } k = 1, 2, \ldots \\
  Q^{(k)} R^{(k)} = A^{(k-1)} \\
  A^{(k)} = R^{(k)} Q^{(k)}
  \]

- With some suitable assumptions, \( A^{(k)} \) converge to Schur form of \( A \) (diagonal if \( A \) is symmetric)
- Similarity transformation of \( A \):

  \[
  A^{(k)} = R^{(k)} Q^{(k)} = \left(Q^{(k)}\right)^T A^{(k-1)} Q^{(k)}
  \]

- But why it works?
Unnormalized Simultaneous Iteration

- To understand QR algorithm, first consider simple algorithm
- Simultaneous iteration is power iteration applied to several vectors
- Start with linearly independent \( \mathbf{v}_1^{(0)}, \cdots, \mathbf{v}_n^{(0)} \)
- We know from power iteration that \( A^k \mathbf{v}_1^{(0)} \) converge to \( \mathbf{q}_1 \)
- With some assumptions, the space \( \langle A^k \mathbf{v}_1^{(0)}, \ldots, A^k \mathbf{v}_n^{(0)} \rangle \) should converge to \( \langle \mathbf{q}_1, \cdots, \mathbf{q}_n \rangle \)
- Notation: Define initial matrix \( \mathbf{V}^{(0)} \) and matrix \( \mathbf{V}^{(k)} \) at step \( k \):

\[
\mathbf{V}^{(0)} = \begin{bmatrix} \mathbf{v}_1^{(0)} | \cdots | \mathbf{v}_n^{(0)} \end{bmatrix}, \quad \mathbf{V}^{(k)} = A^k \mathbf{V}^{(0)} = \begin{bmatrix} \mathbf{v}_1^{(k)} | \cdots | \mathbf{v}_n^{(k)} \end{bmatrix}
\]
Unnormalized Simultaneous Iteration

- Define orthogonal basis for column space of $V^{(k)}$ by reduced QR factorization $\hat{Q}^{(k)} \hat{R}^{(k)} = V^{(k)}$

- We assume that
  1. leading $n+1$ eigenvalues are distinct, and
  2. all leading principal submatrices of $\hat{Q}^T V^{(0)}$ are nonsingular where $\hat{Q} = [q_1|\cdots|q_n]$

- We then have columns of $\hat{Q}^{(k)}$ converge to eigenvectors of $A$:
  \[ \|q_j^{(k)} - (\pm q_j)\| = O(c^k), \]

  where $c = \max_{1 \leq k \leq n} |\lambda_{k+1}|/|\lambda_k|$

- Proof idea: Show that subspace of any leading $j$ columns of $V^{(k)} = A^k V^{(0)}$ converges to subspace of first $j$ eigenvectors of $A$, so does the subspace of any leading $j$ columns of $\hat{Q}^{(k)}$. 