AMS 526 Sample Questions for Final Exam

December 8, 2012

1. (20 points) Answer true or false with a brief justification. (No credit without justification.)

   (a) If $\boldsymbol{A} \in \mathbb{C}^{m \times m}$ is Hermitian positive definite, then the eigenvalues of $\boldsymbol{B}\boldsymbol{A}\boldsymbol{B}^T$ are all real and positive for any nonsingular $\boldsymbol{B} \in \mathbb{C}^{m \times m}$.

   (b) If a nonsingular matrix $\boldsymbol{A} \in \mathbb{C}^{m \times m}$ has a large relative condition number in 2-norm, we may reduce the condition number by left-multiplying an appropriate matrix $\boldsymbol{B} \in \mathbb{C}^{m \times m}$ (i.e., $\kappa_2(\boldsymbol{B}\boldsymbol{A}) < \kappa_2(\boldsymbol{A})$).

   (c) The eigenvalues of a real matrix are not necessarily real but their sum must be real.

   (d) If $\boldsymbol{A} \in \mathbb{C}^{m \times m}$, $\boldsymbol{B} \in \mathbb{C}^{m \times m}$, and $\boldsymbol{A}$ is nonsingular, then $\boldsymbol{A}\boldsymbol{B}$ and $\boldsymbol{B}\boldsymbol{A}$ have the same set of eigenvalues.

   (e) If $\boldsymbol{A}$ is a unitary matrix, then its eigenvalues all have magnitude 1 and it has a full set of orthonormal eigenvectors.

   (f) The conjugate gradient method applies to symmetric indefinite linear systems.

   (g) Reduction to Hessenberg form by Householder reflectors produces tridiagonal matrices for skew Hermitian matrices.

   (h) For symmetric matrices, its singular value decomposition is the same as its Schur factorization.

   (i) If every singular value of a matrix $\boldsymbol{A}$ is zero, then $\boldsymbol{A} = \boldsymbol{0}$.

   (j) If a matrix is normal, then its eigenvalues are all real if and only if it is Hermitian.

2. (10 points) Order the following procedures from the least work required to the most work required, for a non-Hermitian, nonsingular matrix $\boldsymbol{A} \in \mathbb{C}^{m \times m}$ with $m \gg 1$. Justify your answer.

   (a) Gaussian elimination with partial pivoting.

   (b) QR factorization by Householder triangularization (with implicit $\boldsymbol{Q}$).

   (c) Computing all the eigenvalues and eigenvectors by first reducing to Hessenberg form.

   (d) Solving an upper triangular system (assuming $\boldsymbol{A}$ is upper triangular) by back-substitution.

   (e) Computing the inverse of the matrix.

3. (10 points) What method(s) would you choose to solve the following problems? Justify your answer.

   (a) A least-squares problem with a moderate condition number (e.g., 10^4).

   (b) A least-squares problem with a rank-deficient coefficient matrix.

   (c) A very large sparse symmetric positive definite linear system.

   (d) An ill-conditioned linear system with multiple right-hand sides.

   (e) Finding all the eigenvalues for a symmetric tridiagonal matrix.

4. (15 points) Let $\boldsymbol{A}$ be a real symmetric matrix.

   (a) Show that the eigenvalues of $\boldsymbol{A}$ are real.

(b) Argue that the eigenvectors of $\boldsymbol{A}$ are real and orthogonal to each other.

(c) Show that the singular values $\boldsymbol{A}$ are equal to the magnitudes of its eigenvalues.

5. (10 points) Prove or disprove: if a matrix is both triangular and normal, then it must be diagonal.

6. (15 points) In the conjugate gradient method for solving $\boldsymbol{Ax} = \boldsymbol{b}$, show that the subspace spanned by the first m search directions is the same as the Krylov subspace $\mathcal{K}_m = \langle \boldsymbol{b}, \boldsymbol{Ab}, \boldsymbol{A}^2\boldsymbol{b}, \ldots, \boldsymbol{A}^{n-1}\boldsymbol{b} \rangle$.

7. (10 points) Let $\boldsymbol{A} \in \mathbb{C}^{m \times m}$ be a Hermitian matrix and $\boldsymbol{q} \in \mathbb{C}^m$ be a vector with $\|\boldsymbol{q}\|_2 = 1$. Prove or disprove: there exists a unitary matrix $\boldsymbol{Q} \in \mathbb{C}^{m \times m}$ whose first column is $\boldsymbol{q}$, such that $\boldsymbol{Q}^*\boldsymbol{AQ}$ is a tridiagonal matrix.

8. (10 points + 10 bonus points) Let $\boldsymbol{A} \in \mathbb{C}^{m \times m}$, and $\boldsymbol{A} = \boldsymbol{B} + i\boldsymbol{C}$, where $\boldsymbol{B}, \boldsymbol{C} \in \mathbb{R}^{m \times m}$.

   (a) (5 points) Show that $\boldsymbol{A}$ is Hermitian if and only $\boldsymbol{M} = \begin{bmatrix} \boldsymbol{B} & -\boldsymbol{C} \\ \boldsymbol{C} & \boldsymbol{B} \end{bmatrix}$ is symmetric.

   (b) (5 points) Show that if $\boldsymbol{A}$ is Hermitian, then every eigenvalue of $\boldsymbol{A}$ is an eigenvalue of $\boldsymbol{M}$.

   (c) (10 bonus points) Suppose $\boldsymbol{A}$ is Hermitian. Express the eigenvalues and eigenvectors of $\boldsymbol{M}$ in terms of those of $\boldsymbol{A}$.